Proposals for the set of pairwise comparisons

International Symposium on the Analytic Hierarchy Process

Zsombor Szádoczki[1,2,3]

[1]Research Laboratory on Engineering & Management Intelligence
Institute for Computer Science and Control (SZTAKI)

[2]Department of Operations Research and Actuarial Sciences
Corvinus University of Budapest

[3]Co-authors: Sándor Bozóki and Hailemariam Abebe Tekile

December 3-6, 2020

Overview

Overview

- Introduction, incomplete PCMs
- Research question
- Introduction to the used mathematical tools
- Results, some graphs
- Motivational example
- Simulations
- Summary, further research questions

Incomplete Pairwise Comparison Matrices

- In many cases there are missing elements in a PCM.
- Causes: lost of data, the comparison is simple not possible (for instance in sport), the decision maker has no time or willingness to make all the comparisons ($n(n-1)/2$).
- The graph representation of Pairwise Comparison Matrices is a suitable tool for handling these cases.
- The arrangements of the known comparisons is of special importance.

Our assumptions

- We can choose the comparisons, i.e., they are not given a priori.
- The set of comparisons is designed completely before the decision making process.
- We do not have any further prior information, we treat the items to be compared in a symmetric way.

Research question

Research question

For incomplete pairwise comparison matrices, when our assumptions hold, which filling in patterns are recommended to estimate the preferences in the best way?

Our answer

Those that can be represented by (quasi-)regular graphs with minimal diameter.

Basic mathematical definitions

Definitions (Connected, k-regular, k-quasi-regular graphs)

- *Connected graph:* In an undirected graph, two vertices u and v are called connected if the graph contains a path from u to v. A graph is said to be connected if every pair of vertices in the graph is connected.
- *k-regular graph:* A graph is called k-regular if every vertex has k neighbours, which means that the degree of every vertex is k.
- *k-quasi-regular graph:* A graph is called k-quasi-regular if exactly one vertex has degree $k+1$, and all the other vertices have degree k.

Basic mathematical definitions II

Remarks

- The k-regularity basically means that the vertices are not distinguished, there is no particular vertex as, for example, in the case of the star graph. We compare every item to the same number of elements, resulting in a kind of symmetry.
- It is also notable that we would like to avoid the cases when two items are compared only indirectly through a very long path.

The diameter of a graph

Definition (The diameter of a graph)

The diameter (denoted by d) of a graph G is the length of the longest shortest path between any two vertices:

$$d = \max_{u,v \in V(G)} \ell(u, v),$$

where $V(G)$ denotes the set of vertices of G and $\ell(.,.)$ is the graph distance between two vertices, namely the length of the shortest path between them.

The diameter of a graph II

Remarks

- We are interested in the smallest nontrivial values of the diameter.
- The smaller the d parameter, the more stable our system of comparisons is.

Results

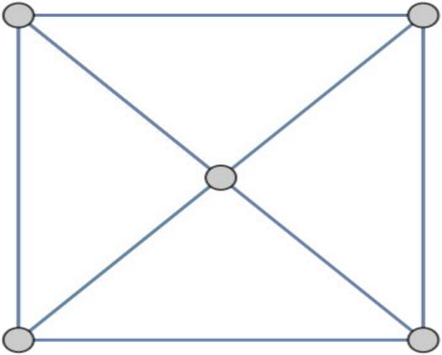
- We provide a list of graphs which shows the patterns of comparisons that are recommended by us.
- We are looking for graphs with minimal diameter for given number of vertices and regularity levels, which has a strong relation with the Degree/Diameter problem known in mathematics.
- In our research we examined the $n = 5, 6, \ldots, 24$ cases, and found that the interesting values for the regularity are $k = 3, 4, 5$ while the interesting values for the diameter of the graph are $d = 2, 3$.
- The completion ratio is defined as follows:

$$c = \begin{cases} \frac{nk/2}{n(n-1)/2} & \text{if } n \text{ or } k \text{ is even} \\ \frac{(nk+1)/2}{n(n-1)/2} & \text{if } n \text{ and } k \text{ are odd} \end{cases}$$

Finding for the different graphs

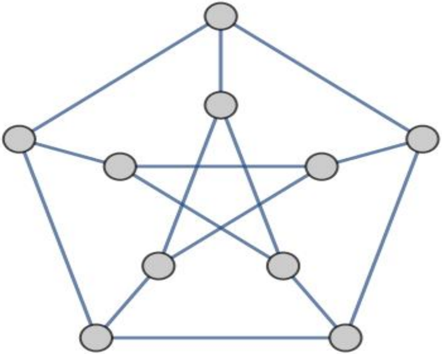
- Already known graphs collected from different articles
- Some other graphs were gained by modifying the already known ones
- A third set was found by the help of computational methods
- A part of them were really challenging and time-consuming to find, we used, different constructing methods, merging and extending other graphs and integer linear programming
- The quasi-regular graphs are our findings

$n = 5, k = 3, d = 2$



- $8/10$ comparisons ($c = 0.8$)
- Unique graph

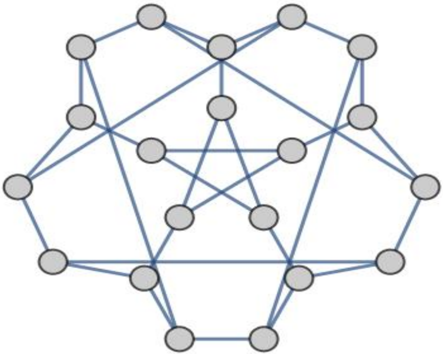
$n = 10, k = 3, d = 2$



Petersen graph

- 15/45 comparisons ($c \approx 0.333$)
- Unique graph

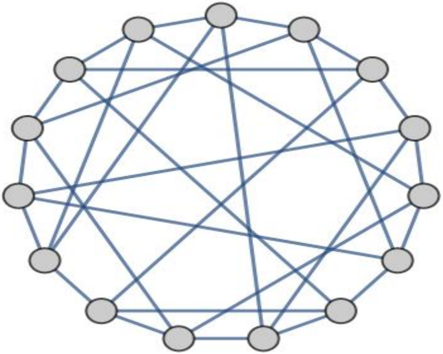
$n = 20, k = 3, d = 3$



(3,3)-graph on 20 vertices ($C_5 * F_4$)

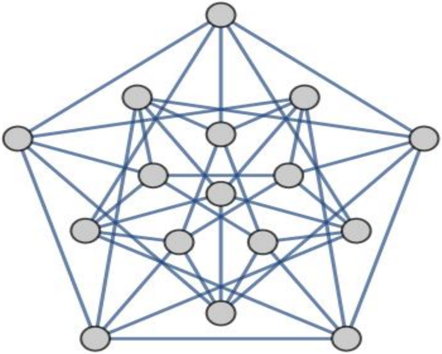
- 30/190 comparisons ($c \approx 0.158$)
- Unique graph

$n = 15, k = 4, d = 2$



- 30/105 comparisons ($c \approx 0.286$)
- Unique graph

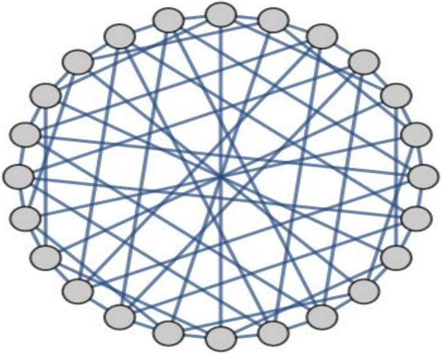
$n = 16, k = 5, d = 2$



Clebsch graph

- 40/120 comparisons ($c \approx 0.333$)
- ≥ 3 graphs

$n = 24, k = 5, d = 2$



$K_3 * X_8$ graph

- 60/276 comparisons ($c \approx 0.217$)
- ≥ 1 graphs

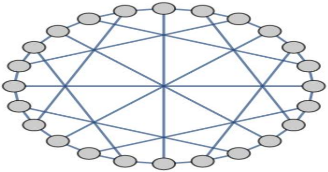
Summarizing table

	k		
n	3	4	5
5	1		
6	2		
7	4		
8	2		
9	2		
10	1		
11	134	37	
12	34	26	
13	353	10	
14	34	1	
15	290	1	
16	14		≥ 3
17	51		≥ 1
18	1		≥ 1
19	≥ 1		≥ 1
20	1		≥ 1
21		≥ 3	≥ 1
22		≥ 1	≥ 1
23		≥ 1	?
24		≥ 1	≥ 1

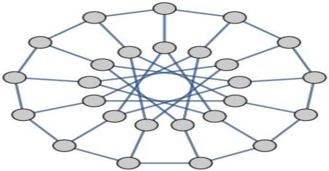
denotes $d = 2$

denotes $d = 3$

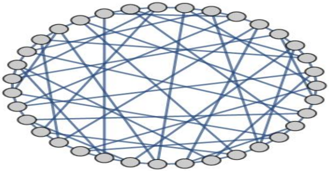
Further graphs



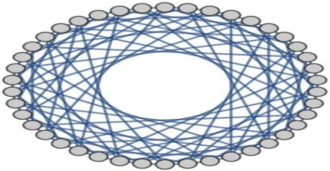
(a) McGee graph
(n=24, k=3, d=4)



(b) Generalized Petersen graph
(n=26, k=3, d=4)

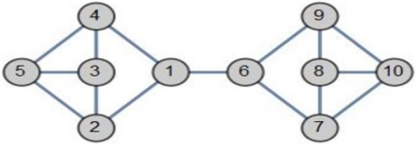


(a) Odd-4 graph
(n=35, k=4, d=3)

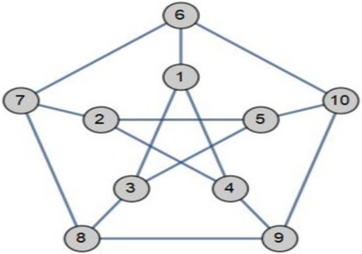


(b) (5,6)-Cage
(n=42, k=5, d=3)

Motivational example



(a) Alternative 3-regular graph



(b) Our recommendation: Petersen graph

The graph representation of two 3-regular designs

Motivational example II

- Weight calculation techniques: Logarithmic Least Squares Method, and Eigenvector Method based on the CR-minimal completion
- Metrics: Euclidean and maximum absolute distance

$$d_{euc}(u,v) = \sqrt{\sum_{i=1}^{n}(u_i - v_i)^2}$$

$$d_{max}(u,v) = \max_{i \in 1 \ldots n} |u_i - v_i|,$$

- With the modification of only one item, the average distances were three times or four times as big in the case of the alternative graph.

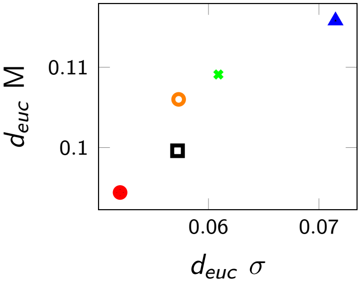
Simulation

Results

- We compared our recommendations with 4 other designs based on the above-mentioned distances and weight calculation methods for 3 different levels of inconsistency.
- These designs are the following: randomly generated graphs, stargraphs, randomly generated graphs of minimal diameter (but not regular), randomly generated regular graphs (but not of minimal diameter).
- The graphs recommended by us resulted in smaller average errors with smaller standard deviations.

Simulation II

$(n = 16, k = 3, d = 3)$



(i) k-(quasi-)regular graphs with minimal diameter
(ii) Random connected graphs
(iii) k-(quasi-)regular, not minimal diameter graphs
(iv) Randomly generated minimal diameter graphs
(v) Modified star graphs with minimal diameter

Summary, future research

- We provided filling in recommendations for incomplete pairwise comparison matrices up until 24 alternatives.
- We presented that both the (quasi-)regularity and the minimal diameter properties are important and needed.
- We used extensive numerical tests that support our results.
- Further research questions: finding larger graphs, testing our recommendations in other models based on pairwise comparisons, studying the average shortest path instead of the diameter.
- If instead of the regularity level, the number of comparisons is given, then what kind of filling in patterns should we recommend? Can we create useful algorithms to answer this question?

Key references

Bozóki, S., Fülöp, J., and Rónyai, L. (2010). On optimal completion of incomplete pairwise comparison matrices. Mathematical and Computer Modelling, 52(1):318 – 333.

Csató, L. (2015). A graph interpretation of the least squares ranking method. Social Choice and Welfare, 44(1):51–69.

Kulakowski, K., Szybowski, J., and Prusak, A. (2019). Towards quantification of incompleteness in the pairwise comparisons methods. International Journal of Approximate Reasoning, 115:221–234.

Miller, M. and Širáň, J. (2013). Moore graphs and beyond: A survey of the degree/diameter problem. Electronic Journal of Combinatorics, Dynamic survey, 20(2):1–92.

Key references

- Pratt, R. W. (1996). The complete catalog of 3-regular, diameter-3 planar graphs.
- Tekile, H. A. (2017) Incomplete Pairwise Comparison Matrices in Multi-Criteria Decision Making and Ranking. MSc Thesis, Central European University.
- Wang, K. and Takahashi, I. (1998). How to select paired comparisons in AHP of incomplete information – strongly regular graph design. Journal of the Operations Research Society of Japan, 41(2):311–328.

Thank You For Your Attention!