RANK PRESERVATION AND RANK STRUCTURE OF JUDGEMENT MATRIX

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ABSTRACT

This paper is written for the rank preservation of evaluating priority weigts and the rank structurein a positive reciprocal or approximate positive reciprocal matrix A that is inconsistent. It will be shown that the satisfying rank structrue of the metrix is an important condition to preserve rank of solutions. Therefore, It is better to first we had better test the rank corretation for the rank structure matrix and second evaluate the priority weights. Besides the rank corretation of matrix A can be checked easily.

INTRODUCTION

We know that many social variables are unmeasurable in soci-system analysis. In order to estimate ther elative priority weights of these variables or objects, many evaluating methods have been advanced. Since systematic variables and objects are unmeasurable, accuracy is not the most impotant factor, however the priority rank orderwill become the basic and stable relationship in the system analysis. Therefore, T.L.Saaty[4] states that for an inconsistent matrix A, rank preservation is one criterion to evaluate which of the priority weight methods is best, and he have shown that the eigenvector method (EM)(Saaty, 1977, 1980) is an asymptotic preserving rank method.

Usually, the judgement matrix $A=(a_{ij})$ can be in following situation,

1. Evey time the pariwise comparison judgement enable to produce the same information about judge's preference as possible.
2. The jugement matrix strives towards positive reciprocal, but it is not positive reciprocal perhaps.
3. If every judgement is independent and thereis no effect between objects, then there must exist no transmission in priority weights.

THE GEOMETRIC DESCRIPTION OF THE JUGEMENT MATRIX

Let us assume that the rows of matrix $A=(a\ \ )$ are elements of the vector space R_n^+,

where $R_n^+ = \{ (a_{i1}, a_{i2}, \ldots, a_{in}) : a_{ij}>0,\ i,j=1,2,\ldots,n\}$,

and the weight voctor subspace W_{n-1}^+,

where $W_{n-1}^+ = \{(w_1, w_2, \ldots, w_n) \mid (w_k) \in R_n^+,\ \sum_{k=1}^{n} w_k = 1\}$

There exists the map $R_n^+ \rightarrow W_{n-1}^+$,

we have

$$w_k = 1 / a_{ik} \sum_{i=1}^{n} \frac{1}{a_{ik}} \qquad k=1,2,\ldots,n. \tag{1}$$

Similarly, assume that the columns of matrix A are elements of the R_n^+, for the map $R_n^+ \rightarrow W_{n-1}^+$, we have

$$w_k = 1 / \sum_{i=1}^{n} a_{ik} \qquad k=1.2, \quad ,n \tag{2}$$

In the following, we take the rows to be the discussion object, and all analysis restuts are appliance to the columns Let a be the ith row vector According to the formula (1), $\alpha' a_i \in R_n^+$ can be considered an element for the weight vector, where $\forall \alpha' \in R_1^+$, so that we project R_n^+ on the hypersurfase S_{n-1}^+ ,

where $S_{n-1}^+ = \{ (a_{i1}, \ldots, a_{in}) \mid (a_{ij}) \in R_n^+, \ |a_i|=1 \}$,

Obviously, the map $S_{n-1}^+ \to W_{n-1}^+$ is smooth. When each element of a has the different rank degree, which is called the unequal priority rank row, the rank vector of a is denoted $\bar{r}_1 = (1, 2, \ldots, n)$; when there exist the same rank degree elements in the row, we denote the rank vector of a_i to be $\bar{r}_i = (1, 1.5, 1.5, 4, \ldots, n)$, for example.

DEFINTION 1. Two row vector a_i. $a_j \in S_{n-1}^+$ have the relation of the same rank order if for each $k, l = 1, 2, \ldots, n$, $a_{ik} > a_{il}$ and $a_{jk} > a_{jl}$, denote $a_i ER a_j$.

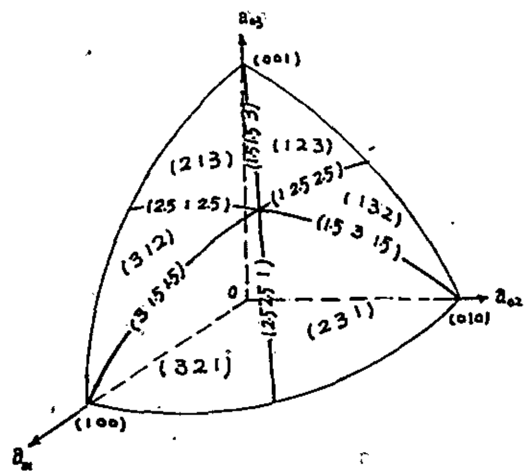
THEOREM 1. Equivalence relation ER determines a uniqe classification S_{n-1}^+ / ER .

PROOF, The same rank relation ER is a rank equivalence relation since ER is a two-ary relation, and (1) if $a_i \in S_{n-1}^+$, then $a_i ER a_i$; (2) if $\forall a_i$, $a_k \in S_{n-1}^+$ and $a_i ER a_k$, then $a_k ER \ a_i$; (3) if $\forall a_i$, a_k, $a_j \in S_{n-1}^+$, $a_i ER a_k$ and $a_k ER a_j$, then $a_i ER a_j$. Certainly, $\forall a_o \in S_{n-1}^+$ must be in an equivalence class at least, for example $\bar{s}_o = \{ a_i \in S_{n-1}^+ \mid a_i ER \ a_o \}$, thus $S_{n-1}^+ = \bigcup \{ \bar{s}_i \mid a_i \in S_{n-1}^+ \}$. On the other hand, suppose two classes be satisfied $\bar{s}_1 \cap \bar{s}_2 \neq 0$, thus at least exsits one vector $a \in \bar{s}_1$ and $a \in \bar{s}_2$, for $\forall a_1 \in \bar{s}_1$, $\exists a_2 \in \bar{s}_2$, $a_1 ER a$ and $a_2 ER a$, so that $a_1 ER a_2$, that is $\bar{s}_1 \subseteq \bar{s}_2$. Similarly, we show $\bar{s}_2 \subseteq \bar{s}_1$, therefore $\bar{S}_1 = \bar{S}_2$. The uniqueness of classification is obvious.

THEOREM 2. On the hypersurface S_{n-1}^+ there exist N! rank equivalence classes in which all elements are unequal priority rank rows.

PROOF, In general, let one of these rank equivalence classes be denoted $\bar{r} = (1, 2, \ldots, n)$, the number of this classes is equal to the number of maps of itself. So that "1" has n images of the map, "2" has (n-1) images, "3" has (n-2) images, and so on. Therefore, the number of these classes is N! .

Now consider the rank geometric structure of the matrix A. The unequal priority rank classes as stated above can be described the (n-1)-dim hyperfaces. It is not difficult to imagin that the classes whose dimension are < (n-1) are between the (n-1)-dim classes , which are called the bounary rank equivalence classes. See Figure 1 which shows n=3. In order to represent the rank correlation degree of two row rank vectors, we define the vectorial angle to be the norm. It ought to be noted that if the angle is equal to zero, then the rank order of two row vectors , must be perfectly correlative ; but rows have perfect rank correlation, the their angles must not be zero.



(Figure 1)

COMPARISON OF EVALUTING METHODS

Many evaluting methods have been put foward as follows,

the eigenvector method (EM), $A \cdot W = \lambda_{max} \cdot W$,

where λ_{max} is the principal eigenvalue of the jugement matrix A;

the least square method (LSM); by minimizing $\sum_{i,j=1}^{n}(a_{ij} - x_j)^2$

we obtain $x_j = \frac{1}{n} \cdot \sum_{i=1}^{n} a_{ij}$ $j=1, 2, \ldots, n$;

the logarithmic least square method (LLSM), by minimizing $\sum_{i,j=1}^{n}(\log a_{ij} - \log x_j)^2$,

we obtain $x_j = (\prod_{i=1}^{n} a_{ij})$ $j=1, 2, \ldots, n$;

where $(x_j) \in R_n^+$. Normalized by (1), LSM and LLSM yield the priority weights. Others, the normalization of the geometric of the rows (NGM)[3], the normalization of the column and sum of the rows (NCM)[1], ect.. It is natural that the priority weights evaluated by these methods not be equal in general.

Consider the following 3×3 judgement matrix

$$\begin{matrix} A' \\ B' \\ C' \end{matrix} \begin{pmatrix} 1 & 4.1 & 1 \\ 1/4 & 1 & 1/4 \\ 1 & 4.1 & 1 \end{pmatrix}$$

See Figure 1, all rows are in one rank equivalunce class (1.5, 3, 1.5). We can directly estimate that the rank of the weight must be (2.5, 1, 2.5), that is $w_{A'} = w_{C'} > w_{B'}$. Applying each method as above, we have ,

	EM	NGM	NCM	LSM	LLSM	
A'	0.4448	0.4448	0.4448	0.4455	0.4453	
B'	0.1103	0.1103	0.1103	0.1090	0.1090	
C'	0.4448	0.4448	0.4448	0.4455	0.4453	$\lambda_{max} \approx 3.0166$

Let us take a symmetric perturbation on the rank class(1.5, 3, 1.5), for example, it produce the following matrix,

$$\begin{matrix} A' \\ B' \\ C' \end{matrix} \begin{pmatrix} 1 & 4 & 1.1 \\ 1/4 & 1 & 1/4 \\ 1.1 & 4 & 1 \end{pmatrix}$$

Though rows of this matrix are not in one rank class, the matrix has the same row rank order and the same column rank order. Similarly, we can estmate that the rank order of weights is (2.5, 1, 2.5), that is $w_{A'} = w_{C'} > w_{B'}$, solutions as follows,

	EM	NGM	NCM	LSM	LLSM	
A'	0.4461	0.4460	0.4460	0.4423	0.4429	
B'	0.1079	0.1080	0.1080	0.1155	0.1143	
C'	0.4461	0.4460	0.4460	0.4423	0.4429	$\lambda_{max} \approx 3.0674$

With a unsymmetric perturbation, we obtain the matrix, for example,

$$\begin{matrix} A' \\ B' \\ C' \end{matrix} \begin{pmatrix} 1 & 4 & 1.1 \\ 0.26 & 1 & 0.25 \\ 1.1 & 4 & 1 \end{pmatrix}$$

At first, we shall estimate that the solutions of EM, NGM and NCM may be chang a little and however the solutions of LSM and LLSM have taken great changed obviosly, as such,

	EM	NGM	NCM	LSM	LLSM	
A'	0.4454	0.4454	0.4454	0.4412	0.4396	
B'	0.1092	0.1092	0.1092	0.1157	0.1150	
C'	0.4454	0.4454	0.4452	0.4431	0.4453	$\lambda_{max} \approx 0.0805$

that is $w_{A'} = w_{C'} > w_{B'}$ and $w_{C'} > w_{A'} > w_{B'}$.

Compare above results, and we note that for a perturbation a variety of methods represent different degrees on sensitivity or stability. It is natural to ask which solutions of methods represent judge's information and which methods can preserve rank?

DEFINITION 2. A method of solution is said to preserve rank if $a_{ik}>a_{il}$, for $i=1,2,\dots,n$, it yields $w_k<w_l$; a method is said to aymptotical preserve rank if as $n\to\infty$, it holds above situation.

THEOREM 3. EM, NGM, NCM, LSM and LLSM preserve rank (Saaty,1984).

PROOF, For a positive reciprocal or approximate positive reciprocal matrix that is inconsistent, from $a_{ik}>a_{il}$ $i=1,2,\dots,n$, we have $a_{kj}<a_{lj}$ $j=1,2,\dots,n$.

For EM we have, $\lambda_{max}w_k=\sum_{j=1}^{n}a_{kj}\cdot w_j\leqslant\sum_{j=1}^{n}a_{lj}\cdot w_j=\lambda_{max}w_l$ $\therefore\ w_k\leqslant w_l$.

For NGM, $\because\ \bar{w}_k=(\prod_{j=1}^{n}a_{kj})^{\frac{1}{n}}\leqslant(\prod_{j=1}^{n}a_{lj})^{\frac{1}{n}}=\bar{w}_l$

and for NCM $\because\ \bar{w}_k=\sum_{j=1}^{n}\frac{a_{kj}}{\sum_{i=1}^{n}a_{ij}}\leqslant\sum_{j=1}^{n}\frac{a_{lj}}{\sum_{i=1}^{n}a_{ij}}=\bar{w}_l$

according to (1), $\therefore\ w_k=\frac{\bar{w}_k}{\sum_{j=1}\bar{w}_j}\leqslant\frac{\bar{w}_l}{\sum_{j=1}\bar{w}_j}=w_l$.

For LSM and LLSM, directly by $a_{ik}>a_{il}$, $i=1,2,\dots,n$,

LSM has $x_k=\frac{1}{n}\cdot\sum_{i=1}^{n}a_{ik}\geqslant\frac{1}{n}\cdot\sum_{i=1}^{n}a_{il}=x_l$, $\therefore\ w_k\leqslant w_l$,

and LLSM has $x_k=(\prod_{i=1}^{n}a_{ik})^{\frac{1}{n}}\geqslant(\prod_{i=1}^{n}a_{il})^{\frac{1}{n}}=x_l$, $\therefore w_k\leqslant w_l$.

COROLLARY. If all rows in one rank equivalence class, then EM, LSM, LLSM, etc. preserve rank.

PROOF, Since all rows have the same rank order, for $i=1,2,\dots,n$, we have $a_{ik}>a_{il}$. So that is obsvious by Theorem 3.

We now develop above result for rank preservation.

DEFINITION 3. If two rank classes $\bar{r}_i$ and $\bar{r}_j$ have the angle $\angle(\bar{r}_i\,\bar{r}_j)=0$, as $n\to\infty$, we say that the rank orders of these classes are perfectly correlative asymptotically.

THEOREM 4. Arbitrary adjacent rank equivalunce classes are perfectly rank correlative asymptotically.

PROOF, In order to prove this result we introduce two terms of the permutation group, commutation and circulantion. Generally we denote one (n-1)dim rank class $\bar{r}=(1,2,\dots,n)$, define the adjacent commutation,

$$\begin{matrix}\bar{r}_0\\ \bar{r}_i\end{matrix}\begin{pmatrix}1,2,\dots,k, & k+1,\dots,n\\ 1,2,\dots,k+1, & k\ \dots,n\end{pmatrix}\qquad 1\leqslant k<n ,$$

define the adjacent circulation,

$$\begin{matrix}\bar{r}_0\\ \bar{r}_j\end{matrix}\begin{pmatrix}1,2,3,\dots,n\\ n,1,2,\dots,n-1\end{pmatrix}\ \text{or}\ \begin{pmatrix}1,2,\dots,n-1,n\\ 2,3,\dots,n\ ,1\end{pmatrix} ,$$

(1) For the adjacent commutation, let the angle of $\bar{r}_0$ and $\bar{r}_i$ be $\alpha=\angle(\bar{r}_0\,\bar{r}_i)$, we have

$$\cos\alpha=\frac{1+2\cdot2+\cdots+k(k+1)+(k+1)k\cdots+n\cdot n}{1^2+2^2+\cdots+n^2}=1-\frac{6}{(n+1)(2n+1)n}$$

as n=9 , $\cos\alpha=0.9965$ $\alpha=4.8$; as $n\to\infty$, then $\cos\alpha=1$, $\therefore\ \alpha\to0$.

and in case of adjacent circulanting transformating, let $\beta=\angle(\bar{r}_0\,\bar{r}_j)$, we have

$$\cos\beta=\frac{1\cdot n+2\cdot1+3\cdot2+\cdots+n(n-1)}{1^2+2^2+\cdots+n^2}=1-\frac{3(n-1)}{(n-1)(2n-1)}$$

as n=9, $\cos\beta=0.8737$, $\therefore\ \beta=29.1$; as $n\to\infty$, then $\cos\beta=1$. $\therefore\ \beta\to0$

(2) Consider the further rank classes from r , which be produced by the k-circulation of $\bar{r}_0$,

$$\begin{matrix} \bar{r}_0 \\ \bar{r}_k \end{matrix} \begin{pmatrix} 1 & ,2 & ,\ldots,k & ,k+1,k+2,\ldots,n \\ n-k+1, & n-k+2, & \ldots,n & ,1 \quad ,2 \quad ,\ldots,n-k \end{pmatrix}$$

where $1\leq k<n$, when k=1, that is an adjacent circulation, let $\gamma=\angle(\bar{r}_0\bar{r}_k)$

$$\cos\gamma=\frac{1(n-k+1)+2(n-k+2)+\ldots+kn+(k+1)1+\ldots n(n-k)}{1^2+2^2+\ldots+n^2}=1-\frac{3k(n-k)}{(n+1)(2n+1)}$$

If and only if k is finite, then as $n\to\infty$, cos =1, $\therefore \gamma\to 0$. Obsviously the bounary rank equivalence classes that are between $\bar{r}_0$ and $\bar{r}_1$ or $\bar{r}_0$ and $\bar{r}_j$ will be perfectly rank correlative asymptotically too.

THEOREM 5. If the rows of the jugement matrix are in the adjacent or near rank classes, EM, NGM, NCM, LSM and LLSM all preserve rank asymptotically.

PROOF, According to Theorem 4, becuase two arbitrary rows are rank correlative asymptotically, there exists N>0, as $n\geq N$, we have $a_{ik}>a_{il}$ for i=1,2,...,n. So that is evident from Theorem 3.

Summarizing as above , we have shown that the property of rank preservation is of the rank structure of matrixes. Therefore, consider the relation of the rank geometric structure of the matrix A to the methods of EM、NGM、NCM、LSM、LLSM、etc.. we have following two results,

1. If all rows of the matrix A is in a rank equivalence class, then the evaluating method as above can be said to preserve rank.
2. If the rows of the matrix A is in some adjacent rank equivalence classes then the evaluating method of solution preserves rank asymptotically.

PROCEDURE OF COMPUTATION

It is clear that the satisfactory rank structure of metrix A is important condition to reserve rank of solutions. At first we had better test the rank correlation of the matrix and adjust the rank structure by consulting with the judge, and second evaluate the priority weights. By this way, we will obtain following advances,

(1) The useful information about judge's favor could be retained by consulting.
(2) The rank preservation of evaluating could be held up.
(3) It could evade recalculation.

Now let introduct Kendall's rank correlation coefficient be the rank corrlation index of the matrix A, here

$$\tau=\frac{\sum_{j=1}^{n}(Rj-1/n\cdot\sum_{j=1}^{n}Rj)}{1/12\cdot n^2(n^2-1)} \qquad 0\leq\tau\leq 1 \qquad (3)$$

where Rj is the sum of the elements in jth column. How large is the satisfactory index value τ ? we assume that the domain of the rank adjacent rows determine the satisfactory correlation index value τ_0 , as following matrix described. See Figure 1, in this matrix one row can be produced from one another by commutating.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ 2 & 1 & 3 & 4 & \ldots & n-1 & n \\ 1 & 3 & 2 & 4 & \ldots & n-1 & n \\ \cdot & & & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & 2 & 3 & 4 & \ldots & n & n-1 \end{pmatrix}$$

by the formula (3).

$$\tau_4 = \frac{2(n^2-n-1)}{1.12n^3(n^2-1)}$$

Let us note that in comparison with the random index (RI), τ_4 is correspond to RI on S_{n-1}^{*}

The following table gives the order of the matrix (first row) and the rank satisfaction index value τ_4:

n:	2	3	4	5	6	7	8	9
τ_4:	0	0.44	0.73	0.85	0.91	0.94	0.96	0.97

Here we give one example considered the following 4×4 judgement matrix, whose row rank structure matrix is on the right,

$$A: \begin{matrix} A' \\ B' \\ C' \\ D' \end{matrix} \begin{pmatrix} 1 & 1/6 & 1/3 & 1/5 \\ 6 & 1 & 4 & 3 \\ 3 & 1/4 & 1 & 4 \\ 5 & 1/4 & 1/4 & 1 \end{pmatrix} \qquad R: \begin{pmatrix} 4 & 1 & 3 & 2 \\ 4 & 1 & 3 & 2 \\ 3 & 1 & 2 & 4 \\ 4 & 1.5 & 1.5 & 3 \end{pmatrix}$$

By (3), we can easily get $\tau = 0.23 < 0.73 = \tau_4$, in comparison with CR=0.158>0.10 [2]
So we ought to consult with judges and adjust the matrix. Assume the new matrix as follows:

$$A: \begin{matrix} A' \\ B' \\ C' \\ D' \end{matrix} \begin{pmatrix} 1 & 1/6 & 1/3 & 1/5 \\ 6 & 1 & 4 & 3 \\ 3 & 1/4 & 1 & 1 \\ 5 & 1/3 & 1 & 1 \end{pmatrix} \qquad R: \begin{pmatrix} 4 & 1 & 3 & 2 \\ 4 & 1 & 3 & 2 \\ 4 & 1 & 2.5 & 2.5 \\ 4 & 1 & 2.5 & 2.5 \end{pmatrix}$$

THus we have $\tau = 0.925 > \tau_4$, comparing with CR=0.039<0.10 . Now let us estimate and obtain solutions,

EM	NGM	NCM	LSM	LLSM
0.0619	0.0612	0.0633	0.0675	0.0612
0.5502	0.5492	0.5450	0.5782	0.5492
0.1733	0.1754	0.1733	0.1598	0.1754
0.2146	0.2142	0.2185	0.1946	0.2142

$\lambda_{max} \approx 4.1053 \qquad \therefore w_{B'} > w_{D'} > w_{C'} > w_{A'}$.

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