# COMPLETENESS AND OPTIMALITY OF AHP BEING APPLIED TO FMADM(MADM)

Wei Shixiao and Huang Jing
East China Institute of Technology
Nanjing, China

ABSTRACT

From the point of view of multiobjective decision making (MODM), the hierarchy structure of AHP is divided into two parts: the hierarcy order combined weight $w$ from total objective level to the last level but one (criterion level), which is regarded as the decision maker's (DM) preference to each criterion, and the hierarchical order odd weight matrix $K(x)$ from the last level (alternative set R) to the criterion level, which is regarded as the decision making matrix $Y(x)$ of multiple attribution decision making(MADM) problem or the optimal membership degree $A(x)$ of fuzzy multiple attribution decision making (FMADM) problem. Thus, AHP is relevant to MADM or FMADM. The completeness and optimal conditions of the algorithm applying AHP to MADM or FMADM are provided in this paper.

MADM model is generally defined by

$$(\text{MADM}) \quad \begin{cases} \max\limits_{x \in R} Y(x) \\ R = (x_1, x_2, \cdots, x_n) \end{cases} \tag{1}$$

where $Y(x) = (y_1(x), \cdots, y_m(x))^T$ is a qualitative or quantitative attribution set of m dimension, and R is a descrete set composed of n alternatives.

Definition 1. Let $\bar{x} \in R$, $\bar{x}$ is said to be an efficient solution of MADM (or EMADM) problem is there does not exist other feasible x (i.e., $x \in R$) such that

$$Y(x) \geqslant Y(\bar{x})^{**}$$

For convenience of notation, we shall denote the set of all efficient solutions of MADM by

$$R^{*}{}_{pa}$$

---

** For arbitrary $Y^1, Y^2 \in R$, $i = 1, \cdot \quad, m$

we have

$$Y^1 \leqq Y^2 \iff y_i^1 \leqslant y_i^2$$

$$Y^1 \leqslant Y^2 \iff Y^1 \leqq Y^2 \text{ and } Y^1 \neq Y^2$$

Definition 2. Let $\bar{x}\in R$. $\bar{x}$ is said to be an absolute optimal solution of MADM (or FMADM) problem if there exists $Y(x)\leqslant Y(\bar{x})$, for any $x$, $\bar{x}$ R. We shall denote the set of all the absolute optimal solutions of MADM (or FMADM) by $R^{*}_{ab}$. $x$ is a strongly optimal solution if

$$Y(x)<Y(\bar{x}) \quad \text{for arbitrary } x,\ \bar{x}\in R$$

Definition 3. Let $\underset{\sim}{B}_1$ be a fuzzy set defined on $[m_1, M_1']$, where $m_1>\inf y_1(x)$, and $M_1=\sup y_1(x)$. $\underset{\sim}{B}_1$ is said to be a fuzzy optimal point set of component $y_1(x)$ of vector $Y(x)$ if $\underset{\sim}{B}_1(M_1)=\bar{c}_1$; where $c_1\in(0, 1]$, and $\underset{\sim}{B}_1(y)$ is a strictly monotone ascending function.

Definition 4. Let $\underset{\sim}{A}_1$ be a fuzzy set definied on the set

$R_1=\{x\in R \mid m_1\leqslant y_1(x)\leqslant M_1$ , $\underset{\sim}{A}_1$ is said to be a fuzzy optimal set of the component $y_1(x)$ of the vector $Y(x)$, if there exists a fuzzy optimal point set $\underset{\sim}{B}_1$ of the component $y_1(x)$ such that $\underset{\sim}{A}_1(x)=\underset{\sim}{B}_1(y_1(x))$.

It is well known that $R^{*}_{ab}=\phi$ for general MADM (or FMADM) problems. To get the satisfactory solution, the preference information must be given by the DM. This means that the mutual "collocation" relation of $y_1(x)$ (or $\underset{\sim}{A}_1(x)$) ($1=1, 2, \ldots, m$, for convenience, denote $1\in M$) must be resolved in MADM (or FMADM). this "collocation" can be completed by some operation among $y_1(x)$ ($1\in M$) or fuzzy sets $\underset{\sim}{A}_1(x)$ ($1\in M$). Therefore, we defined h operator as follows.

Definition 5. Suppose $\underset{\sim}{H}(x)$ be a fuzzy set defined on the set $\bar{R}$ $(\bar{R}=\bigcap_{1=1}^{m} R_1)$ $\underset{\sim}{H}(x)$ is a fuzzy multiple attributive optimal set if there exsits a fuzzy optimal set $\underset{\sim}{A}_1$ of the component $y_1(x)$ ($1\in M$) such that

1. $\underset{\sim}{H}(x)=h(\underset{\sim}{A}(x))=h(\underset{\sim}{A}_1(x), \underset{\sim}{A}_2(x), \ldots, \underset{\sim}{A}_m(x))$ (2)

is a strictly monotone ascending funtion.(**)

2. $h(d,d,\ldots,d)=d$ for arbitrary $d$, $d\in[0,1]$ (3)

The relation between AHP and MADM (or FMADM) is as follows:
1. The hierarchical order combined weight vector $W=[w_1,w_2,\ldots w_m]^T$, which is decided by AHP from total objective to criterion level, corresponds to the weight vector which represents the mutual importance of each component of $Y(x)$ (or $\underset{\sim}{A}(x)$) in MADM (or FMADM). These components of weight vector satisfy the unitized condition, i.e.,

$$\sum_{1=1}^{m} w_1=1 \qquad \text{for } w_1>0,\quad 1\in M \tag{4}$$

---

(**) Real function h which is defined on $E^m$ is said to be a strictly monotone ascending function if for arbitrary $t^1\in E^m$, $t^2\in E^m$, $t^1\leqslant t^2$ implies $h(t^1)<h(t^2)$, h is said to be secondly strict monotone ascending function if $t^1<t^2$ implies $h(t^1)<h(t^2)$.

2. The hierarchical order odd weight matrix $K(x)=[k_1(x),k_2(x),\ldots,k_m(x)]^T$, which is decided by AHP from the last level (alternative set R) to criterion level, corresponds to the decision making matrix $Y(x)=[y_1(x),y_2(x),\ldots,y_m(x)]^T$ in MADM, where $y_l(x)=[y_l(x_1),y_l(x_2),\ldots,y_l(x_n)]$ is the decision making vector of l-th attribution, or corresponds to the fuzzy optimal set $\underset{\sim}{A}(x)=[\underset{\sim}{A}_1(x), \underset{\sim}{A}_2(x), \ldots, \underset{\sim}{A}_m(x)]^T$ in FMADM, where $\underset{\sim}{A}_l(x)=[\underset{\sim}{A}_l(x_1), \underset{\sim}{A}_l(x_2), \ldots, \underset{\sim}{A}_l(x_n)]$ is the optimal membership degree of l-th attribution.

3. In construting the judge matrix $\{a_{ij}^{(l)}\}_{n*n}$ of $K_l(x)$, we assume the following conditions.

Condition 1. The judge matrix is said to be completely consistent if there exist $a_{ij}^{(l)}= a_{ik}^{(l)}/ a_{jk}^{(l)}$ for arbitrary i, j, $k\in N$ and for arbitrary $l\in M$. (5)

Condition 2. The judge matrix is said to be quasi-consistent if $y_l(x_i)>y_l(x_j)$ implies $a_{ik}^{(l)}> a_{jk}^{(l)}$, for arbitrary $i,j,k\in N$, $l\in M$. ------(6)

Theorem 1. The total combined weight $W^TK(x)$ of AHP is the fuzzy optimal set $\underset{\sim}{H}(x)$ of FMADM problem if condition 1 or 2 holds.

Proof: Suppose $y_l(x_j)>y_l(x_k)$ $l\in M$

If the completely consistent condition 1 holds, from the property of consistent matrix[4], we have:

$$a_{jk}=K_l(x_j)/K_l(x_k)>1, \quad \text{so } K_l(x_j) > K_l(x_k).$$

that is $\underset{\sim}{B}_l(y_l(x_j))> \underset{\sim}{B}_l(y_l(x_k))$ and $\underset{\sim}{A}_l(x_j)> \underset{\sim}{A}_l x_k)$.

Hence $B_l(.)$ is a strictly monotone ascending function. If the quasi-consistent condition 2 holds, under the same condition we have [4] $K_l(x_j)>K_l(x_k)$, that is, $\underset{\sim}{B}_l(y_l(x_j))>\underset{\sim}{B}_l(y_l(x_k))$ and $\underset{\sim}{A}_l(x_j)>\underset{\sim}{A}_l(x_k)$. $B_l(.)$ is still a strictly monotone ascending function. We suppose

$\underset{\sim}{A}(x_j)>\underset{\sim}{A}(x_k)$, since $w_l>0$, we have,

$$\sum_{l=1}^{m} w_l K_l(x_j) > \sum_{l=1}^{m} w_l K_l(x_k) \tag{7}$$

$$\sum_{l=1}^{m} w_l K_l(x_j) \hat{=} h(\underset{\sim}{A}_1(x_j),\underset{\sim}{A}_2(x_j),\ldots,\underset{\sim}{A}_m(x_j))=\underset{\sim}{H}(x_j)$$

Therefore, $\underset{\sim}{H}(x)$ is a strictly monotone ascending function of $\underset{\sim}{A}(x)$ where $\underset{\sim}{A}(x)=[\underset{\sim}{A}_1(x),\underset{\sim}{A}_2(x),\ldots,\underset{\sim}{A}_m(x)]^T$.

Furthermore, for arbitrary d, $d\in[0,1]$ there exists

$$\underset{\sim}{H}(d)= d\sum_{l=1}^{m} w_l=d \tag{8}$$

From definition 5, we have $W^TK(x)=\underset{\sim}{H}(x)$. (9)

Thus the total combined weight determined from AHP under the condition 1 or 2 is the optimal membership degree of the FMADM problem. This completes the theorem 1.

Corollary 1. When $K(x)$ is regarded as decision making matrix $Y(x)$ in MADM, the total combined weight of AHP is the total utility function of MADM problem if condition 1 or 2 holds.

Theorem 2. If a judge matrix satisfies condition 1 or 2, the necessary condition for $x_e \in R$ being a strongly optimal solution of FMADM problem is: $\underset{\sim}{H}(x_e) > \underset{\sim}{H}(x)$ for arbitrary $x \in R \setminus x_e$.

Proof: Since $x_e$ is a strongly optimal solution,

$Y(x_e) > Y(x)$ holds, for arbitrary $x \in R \setminus x_e$,

this means that $y_l(x_e) > y_l(x)$ $(l \in M)$, for arbitrary $x \in R \setminus x_e$

for the strictly monotone ascending function $\underset{\sim}{B}(.)$ we have

$$\underset{\sim}{B}_l(y_l(x_e)) > \underset{\sim}{B}_l(y_l(x_i)) \qquad l \in M, \quad i \in N \setminus e,$$

that is $\underset{\sim}{A}_l(x_e) > \underset{\sim}{A}_l(x_i)$, $l \in M$, $i \in N \setminus e$.

From the condition 1 or 2 we have

$$K_l(x_e) > K_l(x_i), \qquad i \in N \setminus e, \qquad l \in M,$$

hence $\underset{\sim}{H}(x_e) = W^T K(x_e) > W^T K(x_i)$ for any $i \in N \setminus e$. So the combined weight of strongly optimal solution is the largest. The proof is complete.

Corollary 2. If the condition 1 or 2 holds, the necessary condition for $x_e \in R$ being a strongly optimal solution of MADM problem is

$U(x_e) = W^T Y(x_e) > W^T Y(x) = U(x)$ for arbitrary $x \in R \setminus x_e$.

Theorem 3. If a judge matrix satisfies condition 1 or 2, the sufficient condition for $x_e$ being an efficient solution of FMADM problem is $\underset{\sim}{H}(x_e) > \underset{\sim}{H}(x)$ for arbitrary $x \in R$.

Proof: If $x_e$ is not an efficient solution, then there exists $x_s \in R$, such that $Y(x_e) < Y(x_s)$, that is,

$$y_l(x_e) \leqslant y_l(x_s) \qquad l \in M,$$

and at least there exists a $p \in M$, such that $y_p(x_e) < y_p(x_s)$.

From the fact of $B_l(.)$ being a strictly monotone ascending function we have

$$\underset{\sim}{B}_l(y_l(x_e)) \leqslant \underset{\sim}{B}_l(y_l(x_s))$$

and

$$\underset{\sim}{B}_p(y_p(x_e)) < B_p(y_p(x_s)) \qquad (l, p \in M)$$

that is, $\underset{\sim}{A}(x_e) \leqslant \underset{\sim}{A}(x_s)$, $K(x_e) \leqslant K(x_s)$,

since $w_l > 0$ $(l \in M)$, we have $\underset{\sim}{H}(x_e) = W^T K(x_e) < W^T K(x_e) = \underset{\sim}{H}(x_s)$,

which contradicts $\underset{\sim}{H}(x_e) > \underset{\sim}{H}(x)$ for arbitrary $x \in R$.

Therefore, $x_e \in R^{*}_{pa}$ . The proof is complete.

We conclude that the alternative with largest total combined weight calculated according to AHP is an efficient solution of FMADM problem.

Corollary 3. If the condition 1 or 2 holds, the sufficient condition of $x_e$ R being an efficient solution of MADM problem

$$U(x_e) = W^T Y(x_e) > W^T Y(x) = U(x) \qquad \text{for any } x \in R.$$

Theorem 4. If the absolute optimal solution set of multiple attribute decision making problem $R^*_{pa} = \phi$, then the efficient solution set $R^*_{pa} = \bar{R}^*_{ab}$ [2]

Proof: From definition 1 and 2, it is obviously $R^*_{pa} \subseteq R^*_{ab}$ .[2]

Now we prove $R^*_{pa} \subseteq R^*_{ab}$ ,

if for arbitrary $\bar{x} \in R^*_{ab}$ then

at least there exists a $x \in R^*_{ab}$ such that $Y(\bar{x}) \leqslant Y(x)$,

but this contradicts the fact $\bar{x} \in R^*_{pa}$ so $\bar{x} \in R^*_{pa}$, this implies $R^*_{pa} \subseteq R^*_{ab}$.

We have $R^*_{pa} = R^*_{ab}$. The proof is complete.

## CONCLUSION

The structure of this method that DM looks for satisfactory solution set and preference information is complete. Under the condition that the judge matrix between alternative level and criterion level is completely consistent or quasi-consistent, the alternative is ordered according to the combined weight, the strongly optimal solution is always arranged at the first place, and the solution at the first place is certainly the efficient solution of MADM (or FMADM) problems. We have applied this method to assignment making for graduates[3], scientific management in some universities[3], and program making for developing science and technology in Jiangsu province. The results were satisfactory consistent judge matrix. Some methods, such as qualitative ordering arrangement, adjacent comparison judgment and replacing the matrix judgment by direct form judgment, have been used to help the specialists in getting consistent judge matrices. Since the completely consistent judgment is a ideal constraint condition, relaxed conditions remain to be studied. Furthermore, studies are needed to relate between quasi-consistence and satisfactory consistence.

## REFERENCES

[1]. Chankong, V.,Haimes, Y.Y., "Multiobjective Decision Making Theory and Methodology", Elsevier Science Publishing Com. Inc. 1983

[2[. Wie Quanling, Wang Righuang, Xu Bing, Wang Jiaying and Bai Wenlin, Mathematical Programming and Optimization Design, National Defence Industry Publishing, Beijing, 1984

[3]. Wei Shixiao, Weapon System Engineering, National Defence Industry Publishing, Beijing, 1988

[4]. Zhao Huanchen, Xushubo and He Jinsheng, The Analytic Hierarchy Process, Science Publishing, Beijing, 1986