OPTIMAL SCALE WITH RESPECT TO CONSISTENCY
IN THE ANALYTIC HIERARCHY PROCESS

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ABSTRACT

From the point of view of psychlogy, T. L. Saaty (1977) suggested using the scale from 1 to 9. which is not consistent, for pairwise comparison in the Analytic Hierarchy Process. For example, suppose we are given three objects, say, A, B, and C. Compare A with B , with the result, say, of 3. Then compare B with C, with the result of 5. Finally compare A with C, the result should be 15 if we would use a consistent scale. Of course, if the number of scale values is finite, then the scale itself will never be consistent as illustrated in the above example. A question arises: Is there an optimal scale such that the consistency index will be minimum for all comparison matrix? This paper has mathematically answered the above question: There is no optimal scale among certain class of scales with respect to consistency in the Analytic Hierarchy Process Therefore the scale from 1 to 9 is a reasonably good one for pairwise comparison in the Analytic Hierarchy Process.

1 Introduction

Saaty (1977) suggested using the following scale for pairwise comparison from the point of view of psychology;

Scale Value	Description
1	Absolutely equal, no difference detected
3	Very slight preference but inconfident in judgment
5	Slight preference detected, confident in judgment
7	Moderate to strong preference, requires little time to detect
9	Large order of magnitude preference, too far apart to adequately scale
2, 4, 6, 8	Intermediate values between the two adjacent judgments

Suppose we have three objects O_1 O_2, and O_3. Compare O_1 with O_2, with the result, say, of 3. Then compare O_2 with O_3, with the result of 5. Finally compare O_1 with O_3, the result should be 15 if we use a consistent scale. Obviously the scale suggested by Saaty is not consistent. Of course, if the number of scale values is finite, then the scale itself will never be consistent as illustrated in the example above. The question arises: Can we do some transformation of the above scale such that the resulting scale is optimal among certain class of scales with respect to consistency? Unfortunately the answer to the above qusetion is no. and we show it in this paper.

2. Definitions and theorems

Instead of the scale $1, 2, \cdots, 9$, we may give the scale $1, 2x, 3x, \cdots, 9x$, where $x > 0$, and see what happens to the consistency of the comparison matrix. Obviously, we do not want to transform the scale value 1, since we desire a ratio scale.

Suppose we get the following two pairwise comparison matrices after the pairwise comparisons are made over the n objects, $O_1, \cdots, O_n$ by using the two scales $1, 2x_0, 3x_0, \cdots, 9x_0$ and $1, 2x, 3x, \cdots, 9x$,

$$A(x_0) = \begin{pmatrix} 1 & a_{12}x_0 & & a_{1n}x_0 \\ \frac{1}{a_{12}x_0} & 1 & & a_{2n}x_0 \\ \cdot & & & \\ \frac{1}{a_{1n}x_n} & \frac{1}{a_{2n}x_n} & & 1 \end{pmatrix}$$

$$A(x) = \begin{pmatrix} 1 & a_{12}x & & a_{1n}x \\ \frac{1}{a_{12}x} & 1 & \cdot & a_{2n}x \\ & & & \vdots \\ \frac{1}{a_{1n}x} & \frac{1}{a_{2n}x} & & 1 \end{pmatrix}$$

where the a_{ij}'s are integers between 1 and 9 for $i < j$. If $a_{ij} = 1$, then define $a_{ij}x$ and $a_{ij}x_0$ to be one. We compute λ_{max} for both matrices where λ_{max} is the Perron root of a positive matrix. We denote the value of λ_{max} for $A(x_0)$ by $\lambda_{max}(x_0)$ and similarly the value of λ_{max} for $A(x)$ by $\lambda_{max}(x)$. Naturally, we wish to find an x_0 that will improve consistency, i.e., will reduce the size of λ_{max}. This gives rise to the following mathematical question: Does there exist an x_0 independent of judgment scale such that

$$\lambda_{max}^{(x_0)} \leq \lambda_{max}^{(x)}$$

for all $x > 0$ and for all a_{ij}? The scale $1, 2x_0, \cdots, 9x_0$ which satisfies the above condition is called the optimal scale in the family of scales $1, 2x, \cdots, 9x$ with respect to consistency. Unfortunately, the optimal scale in the above family does not appear to exist.

Let us first consider the case where there are three objects to be compared. Suppose we get the following matrix after pairwise comparisons are made:

$$\hat{A} = \begin{pmatrix} 1 & a_{12}x & a_{13}x \\ \frac{1}{a_{12}x} & 1 & a_{23}x \\ \frac{1}{a_{13}x} & \frac{1}{a_{23}x} & 1 \end{pmatrix}$$

where a_{12}, a_{13} and a_{23} are integers between 1 and 9, and if $a_{ij} = 1$ for $i < j$ we define $a_{ij}x = 1$.

The characteristic polynomial of the matrix $\hat{A}$ is

$$|\lambda I - \hat{A}| = \begin{vmatrix} \lambda - 1 & -a_{12}x & -a_{13}x \\ -\frac{1}{a_{12}x} & \lambda - 1 & -a_{23}x \\ -\frac{1}{a_{13}x} & -\frac{1}{a_{23}x} & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 1)^3 - 3(\lambda - 1) - \left(\frac{a_{13}}{a_{12}a_{23}x} + \frac{a_{12}a_{23}x}{a_{13}}\right).$$

By solving the characteristic equation of $\hat{A}$, one can get (see Saaty, 1980)

$$\begin{aligned}\lambda^{(x)}_{max} &= \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}x}} + \sqrt[3]{\frac{a_{12}a_{23}x}{a_{13}}} + 1 \\ &= \sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}\, x^{\frac{-1}{3}} + \sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}\, x^{\frac{1}{3}} + 1.\end{aligned}$$

Let us take the derivative of $\lambda^{(x)}_{max}$ with respect to x. We have

$$\begin{aligned}\frac{d\lambda^{(x)}_{max}}{dx} &= \frac{-1}{3}\sqrt[3]{\frac{a_{13}}{a_{12}a_{23}}}\, x^{\frac{-4}{3}} + \frac{1}{3}\sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}\, x^{\frac{-2}{3}} \\ &= \frac{1}{3}\sqrt[3]{\frac{a_{12}a_{23}}{a_{13}}}\, x^{\frac{-2}{3}}\left[1 - x^{\frac{-2}{3}}\left(\frac{a_{13}}{a_{12}a_{23}}\right)^{\frac{2}{3}}\right].\end{aligned}$$

By setting the derivative equal to zero and solving the resulting equation, we obtain the solution

$$x_0 = \frac{a_{13}}{a_{12}a_{23}}.$$

It is easy to see $\frac{d\lambda^{(x)}_{max}}{dx} > 0$ if $x > x_0$ and $\frac{d\lambda^{(x)}_{max}}{dx} < 0$ if $x < x_0$. Hence x_0 is the only minimum point of $\lambda^{(x)}_{max}$. Also notice that at $x = x_0$ we have $\lambda^{(x)}_{max} = 3$, that is, at $x = x_0$, the matrix $\hat{A}$ has perfect consistency.

The minimum point of $\lambda^{(x)}_{max}$, $x_0 = \frac{a_{13}}{a_{12}a_{23}}$, does depend on the judgment scale a_{12}, a_{13} and a_{23}. But we do not know what values a_{12}, a_{13}, and a_{23} have before an experiment. In this case, then, there does not exist an x which is independent of the judgment scales such that the corresponding scale is optimal with respect to consistency.

Now if $\frac{a_{13}}{a_{12}a_{23}} > 1$, then no x in $(0, 1]$ will give a smaller value for $\lambda_{max}(x)$. On the other hand, if $\frac{a_{13}}{a_{12}a_{23}} < 1$, then no value of x in $[1, -\infty)$ will give a smaller value for λ_{max}. Unfortunately, before an experiment we do not know whether $\frac{a_{13}}{a_{12}a_{23}} < 1$ or $\frac{a_{13}}{a_{12}a_{23}} > 1$. Thus, it is impossible to know before an experiment whether $x > 1$ or $x < 1$ will give the smaller $\lambda^{(x)}_{max}$ to improve the consistency.

Let us consider the case where there are four objects to be compared. This case is much more complicated than that in which we have only three objects to be compared. Suppose we have the following matrix after the pairwise comparisons are made:

$$\hat{A} = \begin{pmatrix} 1 & ax & bx & cx \\ \frac{1}{ax} & 1 & dx & ex \\ \frac{1}{bx} & \frac{1}{dx} & 1 & fx \\ \frac{1}{cx} & \frac{1}{ex} & \frac{1}{fx} & 1 \end{pmatrix},$$

where a, b, c, d, e, and f are positive integers between one and nine

The characteristic equation of $\hat{A}$ is

$$\lambda^4 \quad 4\lambda^3 - (\alpha \quad 8)\lambda - (\alpha - 4 \quad 5) = 0$$

where

$$\alpha = \left(\frac{dfx}{e} + \frac{e}{dfx}\right) + \left(\frac{aex}{c} + \frac{c}{aex}\right) + \left(\frac{adx}{b} + \frac{b}{adx}\right) + \left(\frac{bfx}{c} + \frac{c}{bfx}\right)$$

$$\beta = 3 + \left(\frac{adfx^2}{c} + \frac{c}{adfx^2}\right) + \left(\frac{ae}{bf} + \frac{bf}{ae}\right) + \left(\frac{cd}{be} + \frac{be}{cd}\right)$$

By solving the characteristic equation of A, Saaty (1980) gets

$$\lambda_{max}^{(x)} = \frac{2 + \sqrt{\gamma + 4}}{2} + \sqrt{\frac{8 - \gamma}{4} + \frac{\alpha}{2\sqrt{\gamma + 4}}}$$

where

$$\gamma = \{(-8 + \frac{\alpha^2}{2} + 8\beta) + \sqrt{[\frac{-4}{3}(\beta + 3)]^3 + (8 - \frac{\alpha^2}{2} - 8\beta)^2}\}^{\frac{1}{3}} + \{(-8 + \frac{\alpha^2}{2} - 8\beta) - \sqrt{[\frac{-4}{3}(\beta + 3)]^3 + (8 - \frac{\alpha^2}{2} - 8\beta)^2}\}^{\frac{1}{3}}.$$

Notice that

$$\frac{d\lambda_{max}^{(x)}}{dx} = \frac{\partial\lambda_{max}^{(x)}}{\partial\gamma}\frac{d\gamma}{dx} + \frac{\partial\lambda_{max}^{(x)}}{\partial\alpha}\frac{d\alpha}{dx}.$$

One could show the equation below

$$\frac{d\lambda_{max}^{(x)}}{dx} = 0$$

is not a polynomial equation in x. In fact, the equation is so complicated that we do not know how to solve it algebraically.

If we have five or more objects to be compared, then the Abel Theorem tells us that we can not even write down the solution for λ_{max} in closed form. Therefore we have to use another approach to attack the problem.

Suppose we have n objects to be compared. After pairwise comparisons are made, we get the following matrix:

$$A = \begin{pmatrix} 1 & ax & & ax \\ \frac{1}{ax} & 1 & & ax \\ \vdots & & \ddots & \\ \frac{1}{ax} & \frac{1}{ax} & & 1 \end{pmatrix}.$$

i.e., the judgment scale values for object i over j are all the same as long as $i < j$ where a is a positive integer between two and nine.

Obviously, when $ax = 1$, we get perfect consistency and $\lambda_{max}^{(x)} = n$. Thus, at $x_0 = \frac{1}{a}$, we get the minimum of $\lambda_{max}^{(x)}$ which is n. Again, x_0 does depend on the judgment scale a.

Is $x_0 = \frac{1}{a}$ the only one minimum point such that $\lambda_{max}^{(x)} = n$? If there exist other minimum points such that $\lambda_{max}^{(x)} = n$, do they depend on a? Let us find the characteristic polynomial of $\hat{A}$ before we answer the questions.

Define

$$f(n) = |\lambda - \hat{A}| = \begin{vmatrix} \lambda - 1 & -ax & \cdot & -ax \\ \frac{-1}{ax} & \lambda - 1 & & -ax \\ & & & \\ \frac{-1}{ax} & \frac{-1}{ax} & & \lambda - 1 \end{vmatrix}$$

The above determinant is a classical one in linear algebra. By using the elementary properties of a determinant, one could find an algebraic expression for this determinant which is given below:

$$f(n) = \frac{\frac{1}{ax}(\lambda - 1 + ax)^n - ax(\lambda - 1 + \frac{1}{ax})^n}{\frac{1}{ax} - ax}.$$

The above result could be found in many textbooks of linear algebra such as Faddeev(1965).

The characteristic equation of $\hat{A}$ is

$$\frac{\frac{1}{ax}(\lambda - 1 + ax)^n - ax(\lambda - 1 + \frac{1}{ax})^n}{\frac{1}{ax} - ax} = 0.$$

Simplifying, we have

$$\left(\frac{\lambda - 1 + \frac{1}{ax}}{\lambda - 1 + ax}\right)^n = (ax)^{-2}$$

There are n solutions to the above equation, which are

$$\frac{\lambda_k - 1 + \frac{1}{ax}}{\lambda_k - 1 + ax} = (ax)^{\frac{-2}{n}} e^{i\frac{2\pi k}{n}}$$

for $k = 1, 2, \ldots, n$.

Notice that $\lambda_{max}^{(x)} \geq n$, which implies

$$\frac{\lambda_k - 1 + \frac{1}{ax}}{\lambda_k - 1 + ax} \geq 0,$$

hence we have,

$$\frac{\lambda_{max}^{(x)} - 1 + \frac{1}{ax}}{\lambda_{max}^{(x)} - 1 + ax} = (ax)^{\frac{-2}{n}}$$

Finally,

$$\lambda_{max}^{(x)} = 1 - \frac{\frac{1}{ax} - (\frac{1}{ax})^{\frac{2}{n} - 1}}{1 - (\frac{1}{ax})^{\frac{2}{n}}}$$

Notice that $\lambda_{max}^{(x)}$ is a differentiable function of x except at the point $x = \frac{1}{a}$ which is a discontinuous point of $\lambda_{max}^{(x)}$. Also notice

$$\lim_{x \to 0^+} \lambda_{max}^{(x)} = -\infty.$$

$$\lim_{x \to -\infty} \lambda_{max}^{(x)} = -\infty.$$

when n - 2. By the calculus, we know that the minimum points of $\lambda_{max}^{(x)}$ have to be in those points x such that $\frac{d}{dx}\lambda_{max}^{(x)} = 0$ or $x = \frac{1}{a}$

$$\frac{d}{dx}\lambda_{max}^{(x)} = n\frac{[-\frac{1}{ax^2} - (\frac{2}{n} - 1)(\frac{1}{ax})^{\frac{2}{n}-2}\frac{1}{ax^2}][1 - (\frac{1}{ax})^{\frac{2}{n}}]}{[1 - (\frac{1}{ax})^{\frac{2}{n}}]^2}$$
$$- \frac{[\frac{1}{ax} - (\frac{1}{ax})^{\frac{2}{n}-1}]\frac{2}{n}(\frac{1}{ax})^{\frac{2}{n}-1}\frac{1}{ax^2}]}{[1 - (\frac{1}{ax})^{\frac{2}{n}}]^2}$$

Multiplying the numerator above by $\frac{1}{a}$ and setting the result equal to zero, we obtain

$$-\frac{1}{a^2x^2} + (\frac{2}{n} - 1)(\frac{1}{ax})^{\frac{2}{n}}][1 - (\frac{1}{ax})^{\frac{2}{n}} - [\frac{1}{ax} - (\frac{1}{ax})^{\frac{2}{n}-1}]\frac{2}{n}(\frac{1}{ax})^{\frac{2}{n}-1} = 0.$$

If $y = \frac{1}{ax}$, the above equation becomes

$$-y^2 + (\frac{2}{n} - 1)y^{\frac{2}{n}}](1 - y^{\frac{2}{n}}) - (y - y^{\frac{2}{n}-1})\frac{2}{n}y^{\frac{2}{n}+1} = 0.$$

The above equation is independent of a, hence its solutions are independent of a, too. Suppose y_0 is a solution of the above equation, then a solution to $\frac{d}{dx}\lambda_{max}^{(x)} = 0$ is $x_0 = \frac{1}{ay_0}$ which does depend on the judgment scale a.

The following theorem summarizes what we have found so far:

Theorem 1. In the family of scales 1, $2x, \cdots, 9x$, where $x > 0$, there does not exist an x which is independent of the judgment scale such that the $\lambda_{max}^{(x)}$ of the corresponding pairwise comparison matrix always attains its minimum at x. Equivalently there does not exist an x such that the corresponding scale is optimal with respect to consistency in this family.

Let us extend the above theorem to a more general case. Suppose the general transformation of the scale $1, 2, \cdots, 9$, is 1, $f(2,x)$, $f(3,x)$, $f(4,x)$, $f(5,x)$, $f(6,x)$, $f(7,x)$, $f(8,x)$, and $f(9,x)$. The family of scales 1, $2x$, $3x$, $4x$, $5x$, $6x$, $7x$, $8x$, and $9x$ is an example of general transformation of the scale 1, 2, 3, 4, 5, 6, 7, 8, and 9 with $f(a,x) = ax$ where $a = 2, 3, \cdots, 9$. If we set $f(a,x) = a^x$, where $a = 2, 3, 4, \cdots, 9$, we get another family of scale 1, 2^x, 3^x, 4^x, 5^x, 6^x, 7^x, 8^x, and 9^x. We require that $f(a,x)$ be positive. For any fixed x, $f(a,x)$ is a strictly increasing function of a or a strictly decreasing function of a.

In the family of scales 1, $f(2,x), \cdots, f(9,x)$, does there exist an x_0 which is independent of the judgment scale such that $\lambda_{max}^{(x)}$ of the corresponding pairwise comparison matrix always attains its minimum at $x = x_0$? The answer is no. See the following counter example:

Suppose we have n objects to be compared. After pairwise comparisons are made, we get the following comparison matrix:

$$\hat{A} = \begin{pmatrix} 1 & f(a,x) & & f(a,x) \\ \frac{1}{f(a,x)} & 1 & & f(a,x) \\ & & & \\ \frac{1}{f(a,x)} & \frac{1}{f(a,x)} & & 1 \end{pmatrix}$$

i.e. each judgment scale value for object i over j is the same as long as i j where a is a positive integer between two and nine

Then

$$\lambda I - A = \begin{vmatrix} \lambda - 1 & -f(a,x) & f(a,x) \\ \frac{1}{f(a,x)} & \lambda - 1 & -f(a,x) \\ \frac{1}{f(a,x)} & -\frac{1}{f(a,x)} & \lambda - 1 \end{vmatrix}$$

Similarly, we could find

$$|\lambda I - A| = \frac{-f(a,x)(\lambda - 1 + \frac{1}{f(a,x)})^n + \frac{1}{f(a,x)}(\lambda - 1 + f(a,x))^n}{\frac{1}{f(a,x)} - f(a,x)}$$

Setting $|\lambda I - A| = 0$, and solving this equation, we get

$$\lambda_{max}^{(x)} = 1 + \frac{[f(a,x)]^{\frac{2}{n}-1} - f(a,x)}{1 - [f(a,x)]^{\frac{2}{n}}}.$$

The possible discontinuous point of $\lambda_{max}^{(x)}$ is the point x such that $f(a,x) = 1$, but we can make it into continuous point. Let $y = f(a,x)$. Then

$$\lambda_{max}^{(y)} = 1 + \frac{y^{\frac{2}{n}-1} - y}{1 - y^{\frac{2}{n}}}.$$

$$\lim_{y \to 1} \lambda_{max}^{(y)} = 1 + \lim_{y \to 1} \frac{y^{\frac{2}{n}-2} - 1}{1 - y^{\frac{2}{n}}}.$$

By L' Hosptial's rule,

$$\begin{aligned} \lim_{y \to 1} \frac{y^{\frac{2}{n}-2} - 1}{1 - y^{\frac{2}{n}}} &= \lim_{y \to 1} \frac{(\frac{2}{n} - 2)y^{\frac{2}{n}-3}}{-\frac{2}{n}y^{\frac{2}{n}-1}} \\ &= \lim_{y \to 1}(-1 + n)y^{-2} \\ &= n - 1. \end{aligned}$$

Hence

$$\lim_{y \to 1} \lambda_{max}^{(y)} = n.$$

Thus if we define $\lambda_{max}^{(y)} = n$ when $y = 1$, then $\lambda_{max}^{(y)}$ is a continuous function of y for $y \in (0, +\infty)$. Also notice that when $n > 2$,

$$\lim_{y \to 0} \lambda_{max}^{(y)} = +\infty,$$

$$\lim_{y \to +\infty} \lambda_{max}^{(y)} = -\infty.$$

Thus, the minimum of $\lambda_{max}^{(y)}$ has to be achieved at some point inside the interval of $(0, -\infty)$. In fact, when $y = 1$, $\lambda_{max}^{(1)} = n$ is the minimum of $\lambda_{max}^{(y)}$. There could be some other minimum point. In any case, suppose y_0 is a minimum point of $\lambda_{max}^{(y)}$, then the point x_0 such that $y_0 = f(a, x_0)$ is a minimum point of $\lambda_{max}^{(x)}$. Since $f(a, x)$ is a strictly increasing function of a or a strictly decreasing function of a, for a fixed x, if a is changed, x_0 has to be changed, too. Thus x_0 does depend on a Hence we get the following theorem:

Theorem 2. In the family of scales $1, f(2,x), \cdots, f(9,x)$, where $f(a,x)$ is a strictly increasing function of a or a stictly decreasing function of a for a fixed x, then there does not exist an x which is independent of the judgment scale such that $\lambda_{max}^{(x)}$ of the pairwise comparison matrix always gets the minimum at

x. Equivalently there does not exist an x such that the corresponding scale is optimal with respect to consistency in this family.

3. Conclusion

We have shown that there is no optimal scale in these two families of scales with respect to consistency. Why do we choose scale 1, 2, 3, ..., 9? The reason comes from Psychophysics. In his book, The Analytic Hierarchy, Saaty answers this question. One interesting question remains: Is there any optimal scale in a different family of scales from the above two families?

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