CHARACTERIZING PRINCIPAL LEFT-RIGHT EIGENVECTOR RECIPROCITY IN A POSITIVE RECIPROCAL MATRIX

Shubo Xu
Institute of Systems Engineering
Tianjin University
Tianjin, China

1. Introduction

In the Analytic Hierarchy Process (AHP) [1],[2],[3], the priority of elements in the same level of a complete hieirachy is given by the right eigenvector of a reciprocal matrix derived by pairwise comparisons of the elements according to a criterion in the adjacent upper level. Saaty [4] proved that the eigenvector priority method is best in estimating priorities because it is faithful in generating such rank order despite inconsistency in the comparisons.

An important problem in priority setting is how to use the two sides of human experience (dominance and dominanted or large than and smaller than) to obtain a "balanced or reasonable" priority. Mathematically, the problem can be considered as a question of how to develop the matrix of dominance of the matrix or the matrix of dominated, calculate and then synthesize the left and right eigenvectors of the pairwise comparison matrix[6]. First, we must solve the following problem: what relationship is there between left and right eigenvectors of the same reciprocal matrix? Saaty [1, p190] conjectured that the reciprocal property between principal left and right eigenvector components holds if and only if the matrix is consistent for n > 4, where n is the order of the reciprocal matrix. This was shown not to hold by Deturck [7] who gave an answer as to when left and right principal eigenvectors of an inconsistent positive reciprocal n*n matrix are reciprocals for n > 4.

In the second theorem below, we give a necessary and sufficient condition for the reciprocal property to hold between the two principal eigenvectors. We then derive some related results for the reciprocal property and discuss the idea of synthesizing of left and right eigenvectors to obtain a balanced priority.

2. Basic Concepts

Let $A=(a_{ij})$ be a positive, reciprocal, square matrix. Thus

$$a_{ij} > 0 \qquad (i,j=1,2,\ldots,n) \qquad (1)$$

and

$$a_{ij} = 1/a_{ji} \qquad (i,j=1,2,\ldots,n) \qquad (2)$$

A is said to be consistent if the following cardinal relationship holds:

$$a_{ij} = a_{ik} a_{kj} \qquad (i,j,k=1,\ldots,n) \qquad (3)$$

The Perron-Frobenius theorem for a nonnegative irreducible matrices, shows that there exists an essentially unique solution w for the eigenvalue problem:

$$Aw = \lambda_{max} w \qquad (4)$$

where max is the maximum or principal eigenvalue of A, and w is the corresponding principal eigenvector.

Let v be the left eigenvector of A corresponding to max, i.e.

$$A^T v = \lambda_{max} v \qquad (5)$$

v is also unique and positive. A^T is the transpose of A. The following theorem is obvious.

<u>Theorem 1</u> Let A be a positive reciprocal matrix that is consistent and let w be the principal right eigenvector of A corresponding to $\lambda max = n$. Let $v_i = 1/w_i$, then $v = (v_1, v_2, \ldots, v_n)$ is the left eigenvector of A corresponding to $\lambda max = n$.

Proof: Since w is the right eigenvector of A corresponding to $\lambda max = n$, we have:

$$Aw = nw \qquad (6)$$

Because A is consistent:

$$a_{ij} = w_i / w_j \qquad (7)$$

so that

$$\sum_{i=1}^{n} a_{ij}/w_i = \sum_{i=1}^{n} (1/a_{ji})(1/w_i) = \sum_{i=1}^{n} [1/(w_j/w_i)](1/w_i) = \sum_{i=1}^{n} 1/w_j = n/w_j$$

therefore $v_i = (1/w_1, \ldots 1/w_n)^T$ is the eigenvector of $A^T = (a_{ij})$ corresponding to $\lambda max = n$.

Saaty [1, p191] proved that the reciprocal property of corresponding components of principal left and right eigenvector of a positive reciprocal matrix holds for n=3 and showed by the following counter example that this property no longer holds for an inconsistent matrix with n = 4.

$$\begin{pmatrix} 1 & 1/2 & 1/100 & 2 \\ 2 & 1 & 1/3 & 10 \\ 100 & 3 & 1 & 6 \\ 1/2 & 1/10 & 1/6 & 1 \end{pmatrix}$$

$\lambda max = 5.73$, $w = (0.031, 0.142, 0.793, 0.034)^T$

$v = (0.506, 0.075, 0.020, 0.399)^T$

The normalized reciprocal of w is given by

$$(0.461, 0.102, 0.108, 0.419)^T.$$

which is defferent from v.

3. A General Necessary and Sufficient Condition for the Reciprocal Property

Let A and B be the same size matrices. Their elementwise multiplication is known as the Hadamard Product written as

$$(A \circ B) = (a_{ij} \circ b_{ij}) \quad (9)$$

Any positive reciprocal matrix can be uniquely expressed by the Hadamard product as

$$A = W \circ E \quad (10)$$

where $W = (w_i/w_j)$, $w = (w_1, \ldots, w_n)^T$ is the right principal eigenvector, $E = (e_{ij})$, $e_{ij} = a_{ij} w_j / w_i$. It is not difficult to show that

$$Ee = \lambda_{max} e \quad (11)$$

where $e = (1, 1, \ldots, 1)^T$ is the right eigenvector of the matrix E corresponding to the principal eigenvalue λmax. In fact, since w is the right principal eigenvector of A we have

$$Aw = \lambda max\, w \quad (12).$$

i.e.

$$\sum_{j=1}^{n} a_{ij} w_j = \lambda max\, w_i \quad (13)$$

so that,

$$\sum_{j=1}^{n} e_{ij} (w_i/w_j)\, w_j = \lambda max\, w_i \quad (14)$$

Therefore

$$\sum_{j=1}^{n} e_{ij}\, w_i = \lambda max\, w_i \quad (15)$$

and

$$\sum_{j=1}^{n} e_{ij} = \lambda max \quad (16)$$

Hence

$$Ee = \lambda_{max}\, e \tag{17}$$

We now give the main theorem of the paper.

<u>Theorem</u> 2 The reciprocal property between corresponding components of the left and right principal eigenvectors holds if and only if

$$E\, e = E^T e \tag{18}$$

i.e. E/λ_{max} is a doubly-stochastic matrix.

Proof: If $E\, e = E^T e$, then because of (18), we have $E\, e = E^T e = \lambda_{max}\, e$. Suppose that w is the right reciprocal eigenvector of A, then

$$A\, w = \lambda_{max}\, w \tag{19}$$

and

$$A = \hat{W}\, E\, \hat{W}^{-1} \tag{20}$$

where $\hat{W} = \text{diag}\,(w_i)$. So that

$$\hat{W}^{-}E\, \hat{W}^{'}w = \lambda_{max}\, w \tag{21}$$

Let $v = (v_1, \ldots, v_n)$ now be the left principal vector of A, then

$$A^T v = \lambda_{max}\, v \tag{22}$$

and

$$\hat{W}^{'} E^T \hat{W}\, v = \lambda_{max}\, v \tag{23}$$

or

$$E^T \hat{W}\, v = \lambda_{max}\, \hat{W}\, v \tag{24}$$

This shows that $W\, v$ is the right principal eigenvector of E, or the left principal eigenvector of E. Since $E\, e = E^T e = \lambda_{max}\, e$, and the principal eigenvector is unique to within a multiplicative constant, we conclude that

$$\hat{W} v = e \tag{25}$$

i.e. corresponding components of left and right principal eigenvectors are reciprocal.

Conversely, if the reciprocal property (25) holds, and since (22)-(24), we have

$$E^T e = \lambda_{max}\, e \tag{26}$$

But

$$E\ e = \lambda max\ e \tag{27}$$

therefore

$$E\ e = E^T e = \lambda\ max\ e \tag{28}$$

It is obvious that $E/\lambda max$ is a doubly-stochastic matrix. This completes the proof.

As a result of the theorem we have the following corollaries:

<u>Corollary 1</u> The reciprocal property of corresponding components of left and right eigenvectors holds for consistent matrices.

Proof: If A is a consistent matrix, then it has a principal right eigenvector $w = (w_1, \ldots, w_n)$. A can be written in the form:

$$A = (w_i / w_j) \tag{29}$$

or in the form:

$$A = (w_i / w_j) \circ E \tag{30}$$

where $E = e\ e^T$, $e = (1, \ldots, 1)^T$. This shows that the conditions of the theorem are satisfied, and the reciprocal property holds.

<u>Corollary 2</u> The reciprocal property holds for 3 by 3 matrices.

Proof: In fact, a 3 by 3 matrix can be expressed in the form:

$$A = \begin{pmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{pmatrix} \tag{31}$$

It is easy to verify that

$$E = \begin{pmatrix} 1 & a\ b\ c & a\ b\ c \\ a\ b\ c & 1 & a\ b\ c \\ a\ b\ c & a\ b\ c & 1 \end{pmatrix} \tag{32}$$

so that

$$\lambda max\ e = E\ e = E^T e = (1 + a\ b\ c + a\ b\ c)\ e \tag{33}$$

and the corollary follows.

Here is an interesting example of a positive reciprocal matrix for which the reciprocal property holds. Let

$$A = \begin{pmatrix} 1 & 1/2 & 1/2 & 2 & 1/2 \\ 2 & 1 & 8 & 2 & 8 \\ 2 & 1/8 & 1 & 2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 & 1/2 \\ 2 & 1/8 & 2 & 1/2 & 1 \end{pmatrix}$$

The right eigenvector of A is:

$$w = (1/8, 1/2, 1/8, 1/8, 1/8)^T$$

and $\lambda_{max} = 6$, C.R. = 0.2232 > 0.1. We have the perturbed matrix

$$E = \begin{pmatrix} 1 & 2 & 1/2 & 2 & 1/2 \\ 1/2 & 1 & 2 & 1/2 & 2 \\ 2 & 1/2 & 1 & 2 & 1/2 \\ 1/2 & 2 & 1/2 & 1 & 2 \\ 2 & 1/2 & 2 & 1/2 & 1 \end{pmatrix}.$$

Since $Ee = E^Te = \lambda_{max} = 6$, Theorem 2 shows that the reciprocal property holds. The left eigenvector of A is given by

$$v = (1/w_1, 1/w_2, 1/w_3, 1/w_4, 1/w_5)^T =$$
$$= (8, 2, 8, 8, 8)^T$$

After normalization we have

$$v = (0.2353, 0.0588, 0.2353, 0.2353, 0.2353)^T$$

In fact, what determines the existence of the reciprocal property of the right and left eigenvectors is not the index of consistency of the reciprocal matrix but the difference between sums of rows and columns of the perturbed matrix E. When the index of consistency is high, the reciprocal property can also hold.

Following this example, we can consider the general structure of the perturbation matrix E which preserves the reciprocal property. Let $F: R \rightarrow R$, be a mapping of the reals to the reals such that $F(S) = \{1/s : s \in S\}$. We call S the inverse-constant set if $S = F(S)$. For example, $S = \{1, 2, 1/2, 3, 1/3, 1, 1\}$, S is an inverse-constant set because $S = F(S)$. The following theorem gives the general structure of the perturbation matrix E:

<u>Theorem 3</u> In a positive reciprocal matrix A the reciprocal property holds if its perturbation matrix $E = \{e_1, e_2, \ldots, e_n\}$ has the following property:

(1) $\sum_{k=1}^{n} e_{ik} = \sum_{k=1}^{n} e_{jk}$, for $i, j = 1, 2, \ldots, n$

(2) e_i is an inverse-constant set ($i = 1, 2, \ldots, n$).

Proof: Note that the perturbation matrix E is also reciprocal. Because of (1), the sums of any two rows of E are equal. The inverse- constant property of e shows that the sum of any row is equal to the sum of any column of E. Hence A has the property of reciprocal components of the right and left eigenvector based on Theorem 3.

The theorem shows the general principle of preservation of the reciprocal property: the perturbation from consistency in any row should be repeated in all other row.

4. Balanced Priority as a Synthesize of Right and Left Eigenvectors

Saaty [4] has shown that the eigenvalue method gives the real priority of compared alternatives even when the judgment matrix is inconsistent. The problem raised here is how to use the two sides of human experience to obtain a balanced priority, especially, under the condition of large difference between row and column sums of the perturbation matrix E. Note that the row sums of E are always equal to λmax. At least one colunm must be different from λmax, if the reciprocal property does not hold. The results of simulated calculations suggest that the larger the absolute difference between the row sums of E and max, the larger the absolute difference of the corresponding normalized components of right and left eigenvectors. If a large difference exists for the perturbation matrix E, it is necessary to synthesize right and left eigenvectors to obtain a reasonable or balanced priority. In this case, finding the mean of the corresponding normalizing components of the right and left eigenvectors should be a better way than only using one eigenvector. Let us illustrate with an example.

Let the judgment matrix be given by:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1/5 & 7 \\ 1/2 & 1 & 1/4 & 1/5 & 1/6 \\ 1/3 & 4 & 1 & 2 & 1/3 \\ 5 & 5 & 1/2 & 1 & 4 \\ 1/7 & 6 & 3 & 1/4 & 1 \end{pmatrix}$$

which has the normalizing right eigenvector:

$$w = (0.2739,\ 0.0447,\ 0.1695,\ 0.3655,\ 0.1465)^T$$

and $\lambda max = 7.1935$, C.R.= 0.2232. The maximum and minimum row sums of the pertrubation matrix

$$E = \begin{pmatrix} 1 & 0.33 & 1.86 & 0.27 & 3.74 \\ 3.06 & 1 & 0.95 & 1.64 & 0.55 \\ 0.54 & 1.05 & 1 & 4.31 & 0.28 \\ 3.75 & 0.73 & 0.23 & 1 & 1.6 \\ 0.27 & 1.83 & 3.47 & 0.62 & 1 \end{pmatrix}$$

are 8.62 and 4.94, respectively. The maximum absolute difference between λ_{max} and the column sums is equal to 2.75. The normalizated left eigenvector is given by

$$v = (.2586, .0712, .1668, .3580, .1454)^T$$

So that, the balance priority vector should be

$$p = (.2662, .0580, .1681, .3617, .1454)^T$$

5. Conclusion

The structure of the perturbation matrix E is of special interest in the analysis of the rules for deriving priorities from human judgment expressed in terms of pairwise comparisons. On division by λ_{max}, E may be regarded as a row stochastic matrix. Establishing a homomorphic relationship between the set of positive reciprocal matrices and the set of perturbation matrices, is useful for the development of the Analytic Hierarchy Process as a decision making tool. When a row stochastic matrix is also column stochastic, the reciprocal property of corresponding components of left and right eigenvectors holds. To obtain the balanced priority of alternatives, it is proposed to synthesize left and right eigenvectors by taking the mean of corresponding components.

The problem remains as to whether the same result is obtained were the judgments in A elicited in terms of "dominated" than "dominating", and if different then how to combine the two outcomes.

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