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THE ANALYTIC HIERARCHY PROCESS: THE POSSIBILITY THEOREM FOR GROUP DECISION MAKING

Kirti Peniwati
Institute for Management Education and Development
Jl. Menteng Raya 9, Jakarta 10340, Indonesia
Lembaga.PPM36@graha.sprint.com

Abstract: Arrow's Impossibility Theorem says that it is generally impossible to derive a rational group choice from ordinal comparisons made by the individual members. This paper demonstrates that, at appropriate consistency levels, and with the use of judgments on a cardinal scale, the Analytic Hierarchy Process negates Arrow's impossibility. Arrow's conditions are satisfied when aggregation is done at the judgment level when individual judgments are consistent, and at the priority level when they are near consistent.

Introduction

The major problem encountered in social choice is how to aggregate individual preferences into a group preference. Prior to the Analytic Hierarchy Process (Saaty, 1990), attempting to develop a theory to aggregate individual's cardinal preferences was considered as "chasing what cannot be caught" (MacKay, 1980). This is the reason why ordinal group aggregation is problematic, complex, and "procedure dependent." Despite the fact that eliciting ordinal preferences may have some advantages, it oversimplifies the representation of voter preferences. More importantly, aggregating ordinal preferences is subject to the paradox of voting. This paradox, also called the Condorcet effect, occurs when aggregating transitive individual ordinal preferences produces an intransitive group choice.

Kenneth Arrow (1963) won a Nobel prize in 1972 for a proof that relied on this paradox plus other problems associated with ordinal group voting. He showed, in what was later called Arrow's Impossibility Theorem, that it is generally impossible to derive a rational group choice from individual ordinal preferences with more than two alternatives:

> "If we exclude the possibility of interpersonal comparisons of utility, then the only methods of passing from individual tastes to social preferences which will be satisfactory and which will be defined for a wide range of sets of individual orderings are either imposed or dictatorial"

Excluding interpersonal utility means that the social or group preference must depend only on individual preferences of the pair (i.e., independence of irrelevant or external alternatives). Group choice will be considered *satisfactory* if it responds at least not negatively to a change of an individual preference, reflects the collective opinion of the individuals, and provides ranking of the various alternatives. Fishburn (1973) summarized Arrow's conditions, and also those of other theorems of a similar nature by other authors, in four basic axioms as follows:

1. Decisiveness (the procedure must generally produce a group order)
2. Unanimity (if all individuals prefer A to B, so does the group)
3. Independence of irrelevant alternatives
4. No dictator (no single individual determines the group order)

The object of this paper is to demonstrate that, at appropriate consistency levels, the AHP negates Arrow's Impossibility Theorem. The paper begins by formal description of the problem followed by the statement of Arrow's Theorem. It is then shown that aggregating individuals consistent judgments using the AHP fundamental scale, and their priorities when the judgments are near consistent, satisfy Arrow's conditions.

The Problem of Social Choice

A group choice problem involves a set of competing (feasible) elements, called alternatives, and a set of preferences of the individual members of a group. Let:

X be the set of feasible elements, called alternatives;

$\mathcal{X}$ be a set of universal elements which include the set of feasible elements X ($X \subset \mathcal{X}$), all potential elements, and every two-element subset of X;

$\#X$ be the number of elements of X;

m be the number of individuals in the group, and $k = 1,\ldots,m$ be an indexing of individuals.

When two elements are compared according to a property or criterion, we say that a binary comparison with respect to the criterion is made. When a judgment or preference is expressed as a result of a binary comparison, we say that a binary relation is determined. A preference relation on X with respect to a criterion c is symbolized by $\succ_c$ or $\sim_c$. A binary relation $A_i \succ A_j$ means that A_i "is more preferred than" or "dominates" A_j and $A_i \sim A_j$ means A_i is "indifferent" to A_j ($A_i,A_j \in X$). A binary relation with respect to a criterion is elementary, and we can use it either to derive the overall preference of each individual, or as a component to be aggregated across individuals to obtain the binary relation according to the group. Preference can be established with respect to one or several criteria. When the comparison is performed directly with no criterion explicitly specified, as in Arrow's theorem, we shall consider that it is a single criterion problem.

The individuals may or may not be required to express the intensity of their preferences. Arrow's theorem involves aggregating individuals' ordinal preferences, while the AHP requires binary relations on a ratio scale and involves redundancy which gives rise to questions of inconsistency of judgments. When comparing two alternatives A and B, the matrix of pairwise comparisons $\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1 & a \\ \frac{1}{a} & 1 \end{bmatrix} \end{array}$ represents the strength of dominance of A over B by a and of B over A by the reciprocal value of $1/a$. From this matrix, by adding and normalizing the two rows, we obtain the vector w of relative (ratio scale) dominance of A and B given by $\begin{bmatrix} \frac{a}{1+a} \\ \frac{1}{1+a} \end{bmatrix}$. If $a \to \infty$, then the weight of $A \to 1$ and that of $B \to 0$. This is a way of obtaining ordinal preferences from dominance ratios.

The way individuals' values (preferences) are aggregated to produce a group choice is called a social choice function. The function produces a social choice for every possible situation within the domain of the function. A set of conditions that must be satisfied by the social choice function are specified to reflect rationality (the characteristic of an individual) and fairness (the basic premise of democracy). The following proof builds on the formalization on Arrow's Theorem by Fishburn [1972] and is related to the AHP's formalization by Saaty [1990].

Arrow's Impossibility Theorem: Ordinal Aggregation

Arrow assumes that an individual's preference order on a set of alternatives is a weak order. A binary relation $\succ_k$ is a weak order if and only if it is asymmetric and negatively transitive. A binary relation is asymmetric if $A_i \succ A_j$ then not $A_j \succ A_i$, and negatively transitive if not $A_h \succ A_i$ & not $A_i \succ A_j$ then not $A_h \succ A_j$ for $A_h,A_i,A_j \in X$.

Let

D be a set of all individuals' preference relations on X in a specific situation. $D = (D_1,\ldots,D_m)$, where $D_k = \succ_k$, is the ordinal preference of the k^{th} individual on X ($k = 1,\ldots,m$).

$\mathcal{D}$	be a set of all possible individual preferences ($D \subset \mathcal{D}$) that might obtain.
F	be a social function, a mapping of the Cartesian product X x D to the set of all possible subsets of X.
F_D	be an asymmetric binary relation on X associated with F such that if $A_iF_DA_j$ then A_i is selected. More formally, $A_iF_DA_j \Leftrightarrow i \neq j$ and $F(\{A_i,A_j\},D) = \{A_i\}$.

For a given $D \in \mathcal{D}$ and $A_i,A_j \in X$, we define $A_i >_D A_j$ if and only if $A_i \succ_k A_j$ for all $\succ_k$ in D.

Arrow's Impossibility Theorem says that:

If the social function F satisfies the conditions:

C_1	m is a positive integer,
C_2	$\#X \geq 3$,
C_3	$\mathcal{D}$ is a set of m-tuples of weak orders on X and every triple in X is free in $\mathcal{D}$ (i.e., every possible m-tuple of individuals' preferences on every $\{A_h,A_i,A_j\} \in X$ appears in some $D \in \mathcal{D}$, the individual orders are introduced directly into the aggregation process),

then at least one of the following conditions must be false:

C_4	F_D on X is a weak order for every $D \in \mathcal{D}$ (*decisiveness*);
C_5	If $A_i,A_j \in X$, $D \in \mathcal{D}$ and $A_i >_D A_j$ then $A_iF_DA_j$ (*unanimity*);
C_6	If $A_i,A_j \in X$, $D,D' \in \mathcal{D}$ and D on $\{A_i,A_j\}$ equals D' on $\{A_i,A_j\}$, then F_D on $\{A_i,A_j\}$ equals $F_{D'}$ on $\{A_i,A_j\}$ (*independence of irrelevant alternatives*);
C_7	There is no $k \in \{1,\ldots,m\}$ such that $A_i,A_j \in X$, $D \in \mathcal{D}$, $A_i \succ_k A_j \Rightarrow A_iF_DA_j$ (*non-dictatorship*).

The AHP: Aggregation of Consistent Judgments

A choice problem can be modelled as an AHP hierarchy with the goal at the top, and the alternatives at the bottom. The decision criteria, according to which the alternatives are evaluated, are located at the middle level of the hierarchy. How well the alternatives satisfy the decision criteria that contribute the most to the achievement of the goal, determines the relative priority of the alternatives. If the relative importance of the criteria with respect to the goal cannot be assessed easily, or if the alternatives cannot be evaluated easily with respect to the criteria, a hierarchy with multi-level criteria must be constructed. We call the elements directly above the alternatives *covering criteria*. The relative importance of the covering criteria with respect to the goal, which we call their global priority, is needed to evaluate the alternatives. Therefore, for our purpose, we may consider a group choice problem as consisting of three things: a set of (covering) criteria, a set of n feasible alternatives, and a set of the m-tuple binary relations between every pair of alternatives with respect to each criterion. Since we will not be concerned with criteria higher up the hierarchy, we will refer to the *covering criteria* simply as *criteria*. Performing a binary comparison and determining a binary relation between alternatives A_i and $A_j \in X$ with respect to a criterion means that a judgment is made by assigning a real number from the AHP's fundamental scale. The procedure to assign judgments and to derive local weights for the elements in a hierarchy are the same for any cluster in a hierarchy, and so is the procedure to derive global weights for the elements in a level. For this reason, we consider only the process of aggregating individuals' judgments with regard to the relative preferences of the alternatives with respect to each criterion.

Let $a_{ij}^{c,k}$ be the judgment of the k^{th} decision maker comparing A_i and A_j with respect to criterion c.

Definition: $A_i \succ_{c,k} A_j \leftrightarrow a_{ij}^{c,k} > 1$.

Definition: $A_i G_P^C A_j \leftrightarrow G(\{A_i, A_j\}, P^C) = \{A_i\}$ if the group judgment

$P^C(A_i,A_j) = a_{ij}^{C,g} = \left(\prod_{k=1}^{m} a_{ij}^{c,k} \right)^{1/m} > 1$. The group judgment is obtained by taking the geometric average of the preference relation P [Aczél and Saaty, 1983; Aczél and Roberts, 1989].

Definition: Given a group of decision makers, an individual member of the group is said to be a *dictator* if and only if for every pair of alternatives $A_i, A_j \in X$ and any criterion $c \in \{1,\ldots,s\}$, his or her judgment coincides with the group judgment, i.e.,

$$a_{ij}^{C,k^*} = \left(\prod_{k=1}^{m} a_{ij}^{C,k} \right)^{1/m} \quad where \quad k^* \in \{1,\ldots,m\} \quad and \quad a_{ij}^{c,k} \neq a_{ij}^{c,h} \quad k \neq h \ .$$

Definition: A set of judgments $\{a_{ij}^{c,k}\} \in P_k^c$, with its corresponding matrix $A^{c,k}$, is said to define a *consistent order* for the k^{th} individual with respect to a criterion c if the following condition holds: $a_{ij}^{c,k} = a_{ih}^{c,k} \times a_{hj}^{c,k}$ for all $i,j,h \in \{1,\ldots,n\}$ and $i \neq j \neq h$.

Definition: A matrix $A^{c,k}$ is said to be reciprocal if its elements satisfy the relation $a_{ij}^{c,k} = \dfrac{1}{a_{ji}^{c,k}}$ $\forall\, i,j \in \{1,\ldots,n\}$, and the set of binary preferences $\{a_{ij}^{c,k}\}$ is said to satisfy the reciprocal property.

Let:

s be the number of criteria in the hierarchy, and $c = 1,\ldots,s$ indexes these criteria;

$\mathfrak{Z}_c$ be the set of mappings from $X \times X$ to $\mathbb{R}^+$ (the set of positive reals) with respect to a criterion $c \in \{1,\ldots,s\}$. $f: \{1,\ldots,s\} \rightarrow \mathfrak{Z}$ and $P_c^k \in f^k(c)$. P_c assigns a real number $a_{ij}^{c,k}$ for each pair of alternatives and a criterion by each individual. It is assumed that all $a_{ij}^{c,k}$ are independent of $a_{ij}^{c,l}$ where $l \neq k$, $l,k \in \{1,\ldots,m\}$;

P be a set of judgment profiles representing the binary relations on X of the individuals in a specific situation. $P = (P_1^c,\ldots,P_m^c)$, where P_k^c is the set of judgments of the k^{th} individual $k \in \{1,\ldots,m\}$ on every pair of elements $\{A_i,A_j\} \in X$ with respect to every criterion $c = \{1,\ldots,s\}$. It is assumed that each P_k^c defines a consistent order;

$\mathcal{P}$ be a set of all possible individuals' set of judgments ($P \subset \mathcal{P}$) that might obtain;

$A^{c,k}$ be a $n \times n$ reciprocal judgment matrix whose elements are $a_{ij}^{c,k}$; -

G be a social function, a mapping of the Cartesian product $X \times P$ to the set of all possible subsets of X;

G_P be an asymmetric binary relation on X associated with G such that if $A_i G_P A_j$ then A_i is selected. More formally, $A_i G_P A_j \Leftrightarrow A_i \neq A_j$ and $G(\{A_i, A_j\}, P) = \{A_i\}$.

Theorem 1: If the social function G satisfies the conditions:

C_1 m is a positive integer,

C_2 $\#X \geq 3$,

C_3' $\mathcal{P}$ is a set of m-tuples of consistent orders $\{a_{ij}^{c,k}\}$ and every triple in X is free in $\mathcal{P}$,

then

C_4 G_P is a weak order for every $P \subset \mathcal{P}$ (*decisiveness*),

C_5 If $A_i, A_j \subset X$, $P \subset \mathcal{P}$ and $A_i \succ_P A_j$ then $A_i G_P A_j$ (*unanimity*),

C_6 If $A_i, A_j \in X$, $P, P' \in \mathcal{P}$ and P on $\{A_i, A_j\}$ equals P' on $\{A_i, A_j\}$, then G_P on $\{A_i, A_j\}$ equals $G_{P'}$ on $\{A_i, A_j\}$ (*independence of irrelevant alternatives*),

C_7 There is no $k \in \{1,\ldots,m\}$ such that $A_i, A_j \in X$, $P \in \mathcal{P}$, $A_i \succ_k A_j \Rightarrow A_i G_P A_j$ (*non-dictatorship*).

Proof:

C_4 G_P is a weak order for every $P \subset \mathcal{P}$ (*decisiveness*).

We need to prove that G_P is

(1) asymmetric: $A_i G_P A_j \leftrightarrow a_{ij}^{c,P} = \left(\prod_{k=1}^{m} a_{ij}^{c,k}\right)^{1/m} > 1$

We have:

$$a_{ji}^{c,P} = \left(\prod_{k=1}^{m} a_{ji}^{c,k}\right)^{1/m} = \left(\prod_{k=1}^{m} \frac{1}{a_{ij}^{c,k}}\right)^{1/m} = \frac{1}{\left(\prod_{k=1}^{m} a_{ij}^{c,k}\right)^{1/m}} < 1$$

and hence:

$$A_i G_P A_j \rightarrow not \;\; A_j G_P A_i \qquad \forall \;\; A_i, A_j \in X \;\; (asymmetric)$$

and

(2) negatively transitive: $not\ (A_i G_P A_h) \Rightarrow \left[\prod_{k=1}^{m} a_{ih}^{c,k}\right]^{1/m} < 1$

Since $a_{ih}^{c,k} = a_{ij}^{c,k} \times a_{jh}^{c,k}$ then $\left[\prod_{k=1}^{m} a_{ij}^{c,k}\right]^{1/m} < 1$ and $\left[\prod_{k=1}^{m} a_{jh}^{c,k}\right]^{1/m} < 1 \Rightarrow \left[\prod_{k=1}^{m} a_{ih}^{c,k}\right]^{1/m} < 1$

and hence: *If not* $(A_i G_P A_j)$ *and not* $(A_j G_P A_h)$ *then not* $(A_i G_P A_h)$.

C_5 If $A_i, A_j \subset X$, $P \subset \mathcal{P}$ and $A_i >>_P A_j$ then $A_i G_P A_j$ (*unanimity*).

$A_i >_P A_j$ means that the preference relation in P indicates that all individuals prefer A_i to A_j. We have $a_{ij}^{c,P} = \left(\prod_{k=1}^{m} a_{ij}^{c,k}\right)^{1/m}$

and hence $a_{ij}^{c,k} > 1 \quad \forall \; k = \{1,\dots,m\}$ then $a_{ij}^{c,P} > 1$, or $A_i G_P A_j$ holds, $A_i, A_j \in X$.

C_6 If $A_i, A_j \in X$, $P, P' \in \mathcal{P}$ and P on $\{A_i, A_j\}$ equals P' on $\{A_i, A_j\}$, then G_P on $\{A_i, A_j\}$ equals $G_{P'}$ on $\{A_i, A_j\}$ (*independence of irrelevant alternatives*).

Let $P \; on \; \{A_i, A_j\} = a_{ij}^{c,k}$ and $P' \; on \; \{A_i, A_j\} = b_{ij}^{c,k}$.

We have:

$$G(\{A_i, A_j\}, P_c) = \begin{cases} \{A_i\} \; if \; a_{ij}^{c,k} > 1 \\ \{A_j\} \; if \; a_{ij}^{c,k} < 1 \end{cases}$$

$$G(\{A_i, A_j\}, P_c') = \begin{cases} \{A_i\} \; if \; b_{ij}^{c,k} > 1 \\ \{A_j\} \; if \; b_{ij}^{c,k} < 1 \end{cases}$$

and $a_{ij}^{c,k} = b_{ij}^{c,k} \Rightarrow G(\{A_i, A_j\}, P_c) = G(\{A_i, A_j\}, P_c') \Rightarrow G_P = G_{P'}$

C_7 There is no $k \in \{1,\dots,m\}$ such that $A_i, A_j \in X$, $P \in \mathcal{P}$, $A_i \succ_k A_j \Rightarrow A_i G_P A_j$ (*non-dictatorship*)

If k^* is a dictator, where $k^* \in \{1,\dots,m\}$

and hence $a_{ij}^{c,k^*} = \left(\prod_{k=1}^{m} a_{ij}^{c,k}\right)^{1/m}$, then there are two possibilities:

(1) $$a_{ij}^{c,k^*} = a_{ij}^{c,k} \quad \forall \; k \in \{1,\dots,m\},$$

or

(2) $$\left(a_{ij}^{c,k^*}\right)^{m} = \prod_{\substack{k=1 \\ k \neq k^*}}^{m} a_{ij}^{c,k} \times a_{ij}^{c,k^*}$$

from which we have:

$$\left(a_{ij}^{c,k^*}\right)^{m-1} = \prod_{\substack{k=1 \\ k \neq k^*}}^{m} a_{ij}^{c,k} \Rightarrow a_{ij}^{c,k^*} = \left(\prod_{\substack{k=1 \\ k \neq k^*}}^{m} a_{ij}^{c,k}\right)^{\frac{1}{m-1}}.$$

The first possibility contradicts the definition that all judgments are different, i.e.,

$a_{ij}^{c,k} \neq a_{ij}^{c,h}$ for $k \neq h$. The second possibility contradicts the definition that the dictator must belong to the group, $k^* \in \{1,\ldots,m\}$, since the right hand side of the last equation is the group judgment without including the judgment of the k^* individual. Therefore, nondictatorship is satisfied.

■

The AHP: Aggregation of Priorities

In general, judgments are ordinally or cardinally inconsistent (when there are more than two alternatives) and $a_{ij}^{c,k}$ is a perturbation (an approximation) of the real but unknown value of the relative dominance (w_i/w_j) of A_i over A_j.

Definition. A set of judgments $\{a_{ij}^{c,k}\}$ is said to define an ϵ-consistent order if the reciprocal matrix $A^{c,k} = \{a_{ij}^{c,k}\}$ has an eigenvalue λ_{max} such that:

$$Consistency\ Ratio = \frac{Consistency\ Index}{Random\ Index} = \frac{\frac{\lambda_{\max} - n}{n-1}}{Random\ Index} < \varepsilon$$

where $\varepsilon \approx 0.10$. Random Index is the average consistency index of randomly generated reciprocal matrices of the same order from the AHP's fundamental scale. A matrix $A^{c,k}$ whose elements define an ϵ-consistent order is called an ϵ-consistent matrix, such that the ratio of the i^{th} and j^{th} elements of its eigenvector represents the true preference of A_i over A_j. In this case, the relative dominance among the alternatives is obtained by capturing dominance in one step, two steps, three steps, and so on and calculating its limiting value. Saaty and Vargas [1984] have shown that a positive ϵ-consistent matrix (ϵ is small) is p-dominant, indicating that for an ϵ-consistent matrix A, rank is determined in terms of the powers of A. Raising the matrix to powers captures dominance among the alternatives in a number of steps. For large ϵ, ϵ-consistent matrices are asymptotically p-dominant.

Definition: A positive matrix $A^{c,k}$ is said to be p-dominant if there is a p_0 such that for $p \geq p_0$ either $\left(a_{ih}^{c,k}\right)^{(p)} \geq \left(a_{jh}^{c,k}\right)^{(p)}$ or $\left(a_{jh}^{c,k}\right)^{(p)} \geq \left(a_{ih}^{c,k}\right)^{(p)}$ for all $h \in \{1,\ldots,n\}$ for any pair of $\{i,j\}$, where $i,j \in \{1,\ldots,n\}$. $(a_{ih}^{c,k})^{(p)}$ is the (i,h) entry of the matrix $(A^{c,k})^{(p)}$. The matrix is asymptotically p-dominant if $p_0 = \infty$.

An alternative is said to dominate a second alternative in, for example, three steps (i.e., along paths of length three), by first dominating a third alternative then the third alternative dominating the fourth alternative and then the fourth alternative dominating the second. Saaty and Vargas [1984] show that when we take an infinite sequence of dominances along paths of length one, two, three, and so on, and calculate its limiting value, we obtain the principal right eigenvector of the matrix A.

Definition: Individual preference is defined as:

$$A_i \succ_{c,k}^{(p)} A_j \rightarrow \left(a_{ih}^{c,k}\right)^{(p)} > \left(a_{jh}^{c,k}\right)^{(p)}$$

Given a group of decision makers, an individual member of the group is said to be a *dictator* if and only if for every alternative $A_i \in X$ and any criterion $c \in \{1,\ldots,s\}$, the elements of his or her vector of preference coincides with that of the group.

$$A_i H_R^c A_j \rightarrow H(\{A_i,A_j\},\ R^c) = \{A_i\} \text{ if the group preference } \left(\prod_{k=1}^{m} \frac{\left(a_{ih}^{c,k}\right)^{(p^*)}}{\left(a_{jh}^{c,k}\right)^{(p^*)}}\right)^{\frac{1}{m}} > 1$$

where $p^* = \max_k \{p_k\}$.

Let

$\left(A^{c,k}\right)^{(p)}$ be the p^{th} power of the matrix $A^{c,k}$, the result of dominance in p steps and assume that the reciprocal matrix $A^{c,k}$ is ϵ-consistent, and $\left(A^{c,k}\right)^{(p)} = \left\{\left(a_{ij}^{c,k}\right)^{(p)}\right\}$;

R be the set of $\left\{\left(a_{ij}^{c,k}\right)^{(p)}\right\}$ generated from subsets of binary preference relations $P_k^c \subset P$, representing p-step ranking (including the asymptotic ranking) of the alternatives according to the k^{th} individual with respect to criterion c;

$\mathcal{R}$ be the set of all possible sets of $\left\{\left(a_{ij}^{c,k}\right)^{(p)}\right\}$ $(R \in \mathcal{R})$ that might obtain;

W be the set of reciprocal matrix of ratios $W = \left\{\dfrac{a_{ih}^{c,k}}{a_{jh}^{c,k}}\right\}^{(p)}$;

H be a social function, a mapping from $\mathcal{R}$ to X;

H_R be an asymmetric binary relation on X associated with H such that if $A_i H_R A_j$ then A_i is selected. More formally, $A_i H_R A_j \Leftrightarrow A_i \neq A_j$ and $H(\{A_i,A_j\},R) = \{A_i\}$.

$A_i \succ_{c,k} A_j \rightarrow a_{ij} > 1$ applies only when the judgments are consistent, and hence, the dominance of A_i over A_j can be obtained in one step, i.e., from a single judgment $a_{ij}^{c,k}$.

Theorem 2: If the social function H satisfies the conditions:

C_1 m is a positive integer,

C_2 $\#X \geq 3$,

C_3' $\mathcal{R}$ is a set of ε -consistent orders $\left\{\succ_{c,k}^{(p)}\right\}$ and every triple in X is free in $\mathcal{R}$,

then

C_4 $\quad H_R$ is a weak order for every $R \subset \mathcal{R}$ (*decisiveness*),

C_5 $\quad$ If $A_i, A_j \subset X$, $R \subset \mathcal{R}$ and $\; A_i \succ_R A_j \;$ then $A_i H_R A_j$ (*unanimity*),

C_6 $\quad$ If $A_i, A_j \in X$, $R, R' \in \mathcal{R}$ and R on $\{A_i, A_j\}$ equals R' on $\{A_i, A_j\}$, then H_R on $\{A_i, A_j\}$ equals $H_{R'}$ on $\{A_i, A_j\}$ (*independence of irrelevant alternatives*),

C_7 $\quad$ There is no $k \in \{1,\dots,m\}$ such that $A_i, A_j \in X$, $R \in \mathcal{R}$, $A_i \succ_k A_j \Rightarrow A_i H_R A_j$ (*non-dictatorship*).

Proof:

For the k^{th} individual, $\; A_i \succ_{c,k} A_j \rightarrow (a_{ih}^{c,k})^{(p_k)} > (a_{jh}^{c,k})^{(p_k)}$

$$or \quad \frac{(a_{ih}^{c,k})^{(p_k)}}{(a_{jh}^{c,k})^{(p_k)}} > 1 \quad \forall\, h,i,j \in \{1,\dots,n\}, \;\; k \in \{1,\dots,m\}, \;\; c \in \{1,\dots,s\}$$

Since in general $\; p_k \neq p_l \quad \forall\, k,l \in \{1,\dots,m\},\, k \neq l \;$, there exists $\; p^* = \max_k p_k \;$ where for $p \geq p^*$

either $\; \dfrac{\left(a_{ih}^{c,k}\right)^{(p)}}{\left(a_{jh}^{c,k}\right)^{(p)}} \geq 1 \;$ or $\; \dfrac{\left(a_{jh}^{c,k}\right)^{(p)}}{\left(a_{ih}^{c,k}\right)^{(p)}} \geq 1 \;$ for all $k \in \{1,\dots,m\}$.

C_4 $\quad H_R$ is a weak order for every $R \subset \mathcal{R}$ (*decisiveness*).
We need to prove that H_R is

(1) $\quad$ asymmetric: $\; A_i H_R^c A_j \rightarrow \left(\prod_{k=1}^{m} \dfrac{\left(a_{ih}^{c,k}\right)^{(p^*)}}{\left(a_{jh}^{c,k}\right)^{(p^*)}} \right)^{1/m} > 1.$

This follows from:

$$\left(\prod_{k=1}^{m} \frac{\left(a_{jh}^{c,k}\right)^{(p^*)}}{\left(a_{ih}^{c,k}\right)^{(p^*)}} \right)^{\frac{1}{m}} = \left(\prod_{k=1}^{m} \frac{1}{\dfrac{\left(a_{ih}^{c,k}\right)^{(p^*)}}{\left(a_{jh}^{c,k}\right)^{(p^*)}}} \right)^{\frac{1}{m}} = \frac{1}{\left(\prod_{k=1}^{m} \dfrac{\left(a_{ih}^{c,k}\right)^{(P^*)}}{\left(a_{jh}^{c,k}\right)^{(p^*)}} \right)} < 1$$

and hence $\; A_i H_R A_j \rightarrow \textit{not} \; A_j H_R A_i \quad \forall\; A_i, A_j \in X \;$ (*asymmetric*) ;

and

(2) $\quad$ negatively transitive: $\; \textit{not}\ (A_i H_R A_l) \Rightarrow \left[\prod_{k=1}^{m} \dfrac{\left(a_{ih}^{c,k}\right)}{\left(a_{lh}^{c,k}\right)^{(p^*)}} \right]^{\frac{1}{m}} < 1 \,.$

This follows from $\; \dfrac{\left(a_{ih}^{c,k}\right)^{(p \geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p \geq p^*)}} = \dfrac{\left(a_{ih}^{c,k}\right)^{(p \geq p^*)}}{\left(a_{lh}^{c,k}\right)^{(p \geq p^*)}} \times \dfrac{\left(a_{lh}^{c,k}\right)^{(p \geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p \geq p^*)}} \;$ and

$$\left(\prod_{k=1}^{m}\frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}}\right)^{\frac{1}{m}} < 1 \quad and \quad \left(\prod_{k=1}^{m}\frac{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{lh}^{c,k}\right)^{(p\geq p^*)}}\right)^{\frac{1}{m}} < 1 \quad \rightarrow \quad \left(\prod_{k=1}^{m}\frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{lh}^{c,k}\right)^{(p\geq p^*)}}\right)^{\frac{1}{m}} < 1$$

Thus if not $A_i H_R A_j$ and not $A_j H_R A_l$ then $A_l H_R A_i$.

C_5 If $A_i, A_j \subset X$, $R \subset \mathcal{R}$ and $A_i >_R A_j$ then $A_i H_R A_j$ (*unanimity*).

$A_i >_R A_j$ means that the dominance relations in R indicate that all individuals prefer A_i to A_j. Now

$$\frac{\left(a_{ih}^{c,R}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,R}\right)^{(p\geq p^*)}} = \left(\prod_{k=1}^{m}\frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}}\right)^{\frac{1}{m}}$$

Thus if $\dfrac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} > 1 \quad \forall\, k = \{1,\ldots,m\} \quad then \quad \dfrac{\left(a_{ih}^{c,R}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,R}\right)^{(p\geq p^*)}} > 1$

or $A_i H_R A_j$, $A_i, A_j \in X$.

C_6 If $A_i, A_j \in X$, $R, R' \in \mathcal{R}$ and R on $\{A_i, A_j\}$ equals R' on $\{A_i, A_j\}$, then H_R on $\{A_i, A_j\}$ equals $H_{R'}$ on $\{A_i, A_j\}$ (*independence of irrelevant alternatives*).

$$Let\ R\ on\ \{A_i,\ A_j\} = \frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} \quad and \quad R'\ on\ \{A_i,\ A_j\} = \frac{\left(b_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(b_{jh}^{c,k}\right)^{(p\geq p^*)}}$$

We have:

$$H\left(\{A_i,\ A_j\},\ R_c\right) = \begin{cases} \{A_i\}\ if\ \dfrac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} > 1 \\ \{A_j\}\ if\ \dfrac{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}} > 1 \end{cases}$$

$$H\left(\{A_i,\ A_j\},\ R'_c\right) = \begin{cases} \{A_i\}\ if\ \dfrac{\left(b_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(b_{jh}^{c,k}\right)^{(p\geq p^*)}} > 1 \\ \{A_j\}\ if\ \dfrac{\left(b_{jh}^{c,k}\right)^{(p\geq p^*)}}{\left(b_{ih}^{c,k}\right)^{(p\geq p^*)}} > 1 \end{cases}$$

$$\frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} = \frac{\left(b_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(b_{jh}^{c,k}\right)^{(p\geq p^*)}}$$

From which we have:

$$or\ H(\{A_i, A_j\}, R_c) = H(\{A_i, A_j\}, R')\ or\ H_R = H_{R'}\ .$$

C_7 There is no $k \in \{1,\ldots,m\}$ such that $A_i, A_j \in X$, $R \in \mathcal{R}$, $A_i \succ_k A_j \Rightarrow A_i H_R A_j$ (*non-dictatorship*)

If k^* is a dictator, where $k^* \in \{1,\ldots,m\}$, then

$$\frac{\left(a_{ih}^{c,k^*}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k^*}\right)^{(p\geq p^*)}} = \left(\prod_{k=1}^{m} \frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}}\right)^{\frac{1}{m}}$$

There are two possibilities:

(1) The p-step dominance according to the k^{th} individual is the same as those of the other individuals:

$$\frac{\left(a_{ih}^{c,k^*}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k^*}\right)^{(p\geq p^*)}} = \frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} \qquad k,k^* \in \{1,\ldots,m\}, \quad k \neq k^*$$

(2) Algebraic manipulation yields:

$$\left(\frac{\left(a_{ih}^{c,k^*}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k^*}\right)^{(p\geq p^*)}}\right)^{m} = \prod_{\substack{k=1\\k\neq k^*}}^{m} \frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} \times \frac{\left(a_{ih}^{c,k^*}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k^*}\right)^{(p\geq p^*)}}$$

$$\left(\frac{\left(a_{ih}^{c,k^*}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k^*}\right)^{(p\geq p^*)}}\right)^{m-1} = \prod_{\substack{k=1\\k\neq k^*}}^{m} \frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} \Rightarrow \frac{\left(a_{ih}^{c,k^*}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k^*}\right)^{(p\geq p^*)}} = \left(\prod_{\substack{k=1\\k\neq k^*}}^{m} \frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}}\right)^{\frac{1}{m-1}} \qquad (*)$$

The first possibility contradicts the definition that all judgments are different, which lead to different p-step judgments:

$$a_{ij}^{c,k} \neq a_{ij}^{c,l} \rightarrow \frac{\left(a_{ih}^{c,k}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,k}\right)^{(p\geq p^*)}} \neq \frac{\left(a_{ih}^{c,l}\right)^{(p\geq p^*)}}{\left(a_{jh}^{c,l}\right)^{(p\geq p^*)}} \qquad \forall\ k,l \in 1,\ldots,m, \quad k\neq l$$

The second possibility contradicts the definition that the dictator must belong to the group, $k^* \in \{1,\ldots,m\}$ since the right hand side of (*) is the group judgment without including the judgment of the k^* individual. Therefore, nondictatorship is satisfied.

■

In the case where matrix $A^{c,k}$ is inconsistent or ϵ-consistent but ϵ is large, dominance among the alternatives can still be obtained in the limit where:

$$\lim_{p\to\infty}\left(a_{ih}^{c,k}\right)^{(p)} = \lim_{p\to\infty}\left(a_{il}^{c,k}\right)^{(p)} \quad \forall\ i,l,h \in \{1,\ldots,n\},\ l\neq h$$

and hence normalizing the matrix $\left(A^{c,k}\right)^{(p\to\infty)}$ by dividing each of its elements by the total of its column, we obtain a matrix whose columns are identical and coincide with the principal eigenvector of $A^{c,k}$ [Saaty, 1990]. The pairwise cardinal dominance according to the k^{th} individual is now given by:

$$\frac{w_i^{c,k}}{w_j^{c,k}} = \frac{\lim\limits_{p\to\infty} \dfrac{\left(a_{ih}^{c,k}\right)^{(p)}}{\sum_{i=1}^{n}\left(a_{ih}^{c,k}\right)^{(p)}}}{\lim\limits_{p\to\infty} \dfrac{\left(a_{jh}^{c,k}\right)^{(p)}}{\sum_{j=1}^{n}\left(a_{jh}^{c,k}\right)^{(p)}}}$$

and their group aggregation is given by:

$$\frac{w_i^c}{w_j^c} = \left(\prod_{k=1}^{m}\frac{w_i^{c,k}}{w_j^{c,k}}\right)^{\frac{1}{m}}$$

As before, the individual preferences and their aggregation rule are given by:

$$A_i \succ_{c,k}^{(p\to\infty)} A_j \ \leftrightarrow\ \frac{w_i^{c,k}}{w_j^{c,k}} > 1 \quad and \quad A_i H_R^c A_j \ \leftrightarrow\ \left(\prod_{k=1}^{m}\frac{w_i^{c,k}}{w_j^{c,k}}\right)^{\frac{1}{m}} > 1$$

■

The following corollary can be proven following the same steps as in Theorem 2.

<u>Definition:</u> A set of judgments $\left\{a_{ij}^{c,k}\right\}$ is said to define an inconsistent order if the matrix $A^{c,k} = \left\{a_{ij}^{c,k}\right\}$ has a Consistency Ratio > 0.10.

<u>Corollary 1</u>: If the social function I satisfies the conditions:

C_1 m is a positive integer,

C_2 $\#X \geq 3$,

C_3' $\mathcal{R}$ is a set of inconsistent orders $\left\{\succ_{c,k}^{(p)}\right\}$ and every triple in X is free in $\mathcal{R}$,

then

C_4 I_R is a weak order for every $R \subset \mathcal{R}$ (*decisiveness*),

C_5 If $A_i, A_j \subset X$, $R \subset \mathcal{R}$ and $A_i \succ_R A_j$ then $A_i I_R A_j$ (*unanimity*),

C_6 If $A_i, A_j \in X$, $R, R' \in \mathcal{R}$ and R on $\{A_i, A_j\}$ equals R' on $\{A_i, A_j\}$, then I_R on $\{A_i, A_j\}$ equals $I_{R'}$ on $\{A_i, A_j\}$ (*independence of irrelevant alternatives*),

C_7 There is no $k \in \{1,\ldots,m\}$ such that $A_i, A_j \in X$, $R \in \mathcal{R}$, $A_i \succ_k A_j \Rightarrow A_i I_R A_j$ (*non-dictatorship*).

Corollary 2: From Theorem 2 and Corollary 1 it follows that $w_i^c \propto \left(\prod_{k=1}^{m} w_i^{c,k}\right)^{\frac{1}{m}}$.

Conclusion

We have shown that Arrow's Impossibility Theorem does not hold when individual preferences are cardinal rather than ordinal. In addition, we also found that when the vectors of priorities for the individuals are known, a way of combining them to represent the group priority, consistent with the propositions of Arrow's Theorem, is the geometric average of the individual priorities.

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