Information Synthesis of Principal Right and Left Eigenvectors

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Abstract

This paper proposes a method which constructs a matrix B which synthesizes the direct and indirect information between two alternatives. By virtue of B, we show that the eigenvector priority truly represents the rank order of alternatives if the corresponding components of principal right and left eigenvectors are reciprocal. In this case, the consistency check becomes meaningless. If the reciprocal property does not hold, the eigenvector priority .... A method to revise the priority is also given.

1. Introduction

In the AHP, a judgment matrix A gives the information of direct pairwise comparisons of n alternatives, but does not give the indirect information between any two alternatives. A^2 provides the direct comparison information between alternatives i and j and indirect information of comparisons between alternatives i and j through k. Therefore A^2 contains more information than A. Similarly A^k has such a property. A^k, and hence, $A^k e/e^T A^k e$ [4], holds synthesized information of direct and indirect comparisons between alternatives i and j. Thus, it seems reasonable to synthesize the information contained in the matrices A, A^2, ..., A^k to determine the rank of the alternatives by the sum:

$$\frac{1}{k}\left(A + \frac{A^2}{\lambda} + \ldots + \frac{A^k}{\lambda^{k-1}}\right)$$

instead of using the matrix A, where λ is the principal eigenvalue of A. In the first theorem below we prove that the limit of the sum exists as k tends to infinity.

In a priority setting people hope to use two sides of human experience (dominance and dominated, or larger than and smaller than) to obtain a reasonable priority [2]. Mathematically, the problem can be considered as a question of how to: develop the dominance matrix or the dominated matrix, and then synthesize the left and right eigenvectors of the pairwise comparisons matrix. However, first we must solve the problem: What is the relationship between principal left and right eigenvectors?

2. Matrix B of Synthesizing Information

Theorem 1. The limit

$$B = \lim_{k \to \infty} \frac{1}{k} \left(A + \frac{A^2}{\lambda} + \dots + \frac{A^k}{\lambda^{k-1}}\right)$$

exists and $B = \lambda wv$, where w, v are principal right and left eigenvectors of A, respectively.

Proof: According to the conclusion in reference [1, p.77] we have:

$$\lim_{k \to \infty} \frac{A^k}{\lambda^k} = wv$$

where $w > 0$, $v > 0$ are the right and left eigenvectors corresponding to λ, respectively, and λ is the maximum eigenvalue of A. Hence, we have

$$\lim_{k \to \infty} \frac{A^k}{\lambda^{k-1}} = \lambda wv$$

Also, it is known that if a sequence a_n converges to a limit a, as n tends to infinity, then the series

$$\frac{1}{n} \sum_{k=1}^{n} a_k \to a \quad as\ n \to \infty.$$

Hence, we have:

$$B = \lambda wv = \begin{vmatrix} w_1v_1 & \dots & w_1v_n \\ w_2v_1 & \dots & w_2v_n \\ \dots & \dots & \dots \\ w_1v_1 & \dots & w_1v_n \end{vmatrix}$$

and the result follows.

The matrix B includes the information of all direct and indirect comparisons between alternatives.

3. Properties of B

Property 1: If A is consistent, then B = A and the corresponding components of principal left and right eigenvectors of A are reciprocal.

Proof: It follows from [1] where it was shown that if A is consistent then $\lambda = n$, $A^k = n^{k-1}A$, and from Theorem 1.

Property 2: $AB = \lambda B$ and $BA = \lambda B$.

Proof : Using Theorem 1 we have:

$$\lim_{k\to\infty} \frac{A^k}{\lambda^{k-1}} = \frac{A}{\lambda} \lim_{k\to\infty} \frac{A^{k-1}}{\lambda^{k-2}} = \frac{A}{\lambda} B \; .$$

and hence $AB = \lambda B$. The proof for $BA = \lambda B$ is similar.

Property 3: $B^2 = \lambda B$, where λ is an eigenvalue of B, and the trace of B is equal to λ.

Proof: It follows from Theorem 1.

Property 4: The rank of B is 1.

4. Consistency of A and Reciprocity of principal left and right eigenvector components.

Saaty [2,p191] proved that the reciprocal property of corresponding components of principal left and right eigenvectors of a positive reciprocal matrix holds for n=3 and conjectured through an example that this property will not hold for an inconsistent matrix with n=4. He conjectured that the reciprocal property between the left and right principal eigenvector components holds if and only if the matrix is consistent for $n \geq 4$, where n is the order of the reciprocal matrix. We will show that this conjecture does not hold.

Theorem 2. The reciprocal property between the left and right principal eigenvector components holds if and only if the diagonal elements of B are given by:

$$w_1 v_1 = \ldots = w_n v_n = \lambda/n.$$

This theorem follows from Property 3 and the property that the vectors **w** and **v** are unique within a multiplicative constant. This theorem shows that the principal left and right eigenvectore components are reciprocal only if the inconsistency is uniformly distributed over the diagonal elements of B, independently of the magnitude of the deviation of A from consistency. From Saaty's point of view [5] we obtain the result that the priority derived by the eigenvector method is the real rank of the alternatives if the corresponding components of the principal left and right eigenvectors are reciprocal.

5. Error of Eigenvector Priority

Let us assume that the diagonal elements of B are not the same. That is,

$$w_1 v_1 = \alpha_1, \; w_2 v_2 = \alpha_2, \; \ldots \; , \; w_n v_n = \alpha_n$$

According to Property 3 and Property 4 we would have:

$$w_1v_1 + \ldots + w_nv_n = \lambda$$

We define the error of w_i as given by:

$$\Delta_i = |w_iv_i - \lambda/n| = |\alpha_i - \lambda/n|, \quad i=1,2,\ldots,n.$$

Thus, the total error of **w** would be given by:

$$\Delta = \Delta_1 + \Delta_2 + \ldots + \Delta_n$$

which represents the deviation of the eigenvector from the reciprocal property.

6. Synthesis of Left and Right Eigenvectors

Using the error analysis we propose the following method to revise the priorities to make the error equal to zero. Let

$$w_i^* = w_i/\alpha_i^{1/2}, \quad v_i^* = v_i/\alpha_i^{1/2} \quad i=1,2,\ldots,n.$$

By construction, the new priorities satisfy $\Delta = 0$, and they are reciprocal of each other.

7. Finding B

The matrix B is unique under the limit sense and it is difficult to obtain. Fortunately the principal right or left eigenvector is unique within a multiplicative constant and thus, we can calculate **w** and **v** by and take (λ/b)**wv** as B, where

$$b = w_1v_1 + \ldots + w_nv_n.$$

References

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