THE EXSISTENCES AND THE EXPRESSIONS OF LIMIT IMPACT PRIORITY AND LIMIT ABSOLUTE PRIORITY

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ABSTRACT

In the setting priority of system with feedback, the synthetical priority which is fit for pure recussing hierarchic structure is invalid. In this case, the limit Impact Priority (LIP) and Limit Absolute Priority (LAP) are considered. This paper simplifies the computing formuli of LIP and LAP. In terms of the definitions of mean Limit Impact Priority (mLIP) and mean Limit Absolute Priority (mLAP), the computing forumuli of mLIP and mLAP are given out.

I. Introduction

The supermatrix is an important tool in the priority setting of system with feedback.[1],[2] Let n be the number of elements in the system and W_{ij} $(i,j=1,2,\ldots,n)$ be the direct impact of ith element on jth element. The supermatrix can be expressed as $W=(w_{ij})_{nxn}$. Let $W_{ij}^{(k)}$ be the impact of ith element on jth element through k steps. Analagous to the characteristic of the state transition matrix of definte homogenous Markor chains, there are the following equations

$$w_{ij}^{(1)} = w_{ij}$$

$$w_{ij}^{(2)} = \sum_m w_{im} w_{mj}$$

$$\vdots \tag{1-1}$$

$$w_{ij}^{(k+h)} = \sum_m w_{im}^{(h)} w_{mj}^{(k)}$$

The equation (1-1) can be expressed in matrix as follows

$$w^{(k+h)} = w^k w^h \tag{1-2}$$

Suppose that the initial priority of ith element is $v_i^{(0)}$ $(i=1,\ldots,n)$ the absolute priority of ith element through k step is defined as

$$V_i^{(k)} = \sum_j w_{ij}^{(k)} V_j^{(0)} \quad ,$$

We especially interest in the Limits of the equations (1-1) and (1-3). They are defined respectively as the limit impact priority of ith element on jth element and the Limit absolute priority of ith element if they exist.
In section 2 the classification of system with feedback is studied; In section 3, the existences and the expressions of LIP and LAP of system with feedback in terms of the classification are discussed in details.

2. The Classification of System with Feedback

As wellknown, a irreducible primitive matrix W is coqredient to the following form

$$\left|\begin{array}{cccccc} 0 & 0 & 0 & 0 & W_c \\ W_1 & 0 & 0 & 0 & 0 \\ 0 & W_2 & & 0 & 0 \\ \cdot & \cdot\ \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & W_{c-1} & 0 \end{array}\right| \qquad (2\text{-}1)$$

Where W_i $(i=1,\dots,n)$ is a submatrix and the zero blocks along diagonal are all square ; furthermore, if let $B_j=W_{j+c-1}\dots W_j$ $(j=1,\dots,c,$ indices taken mudulo C, i.e. $W_{j+c}=W_j)$. The matrix B_j is primitive for every $j=1,\dots,n$.
The form (2-1) is called the standard form of irreducible matrix. When C=1, W is primitive; when $C>=2$, W is circular, C is called the period of matrix W. If the supermatrix W of a system with feedback is circular with period C, the system is called circular with period C.
Any supermatrix of a system with feedback is coqredient to the following form[1]

$$W=\left|\begin{array}{ccccc} A_{11} & & & 0 & B_1 \\ & A_{22} & & & B_2 \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot \\ 0 & & & A_{kk} & B_k \\ & 0 & & & Q \end{array}\right| \qquad (2\text{-}2)$$

Where A_{ii} $(i=1,2,\dots k)$ is irreducible and Q is a nonegative square matrix of which spectrum radium is less than 1. B_i $(i=1,\dots,k)$ is a corresponding nonegative matrix . Especially, there may not be final column and final row blocks, i.e.

$$W=\left|\begin{array}{cccc} A_{11} & & & 0 \\ & A_{22} & & \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 0 & & & A_{kk} \end{array}\right| \qquad (2\text{-}3)$$

The equation (2-2) or (2-3) is called reducible standared form of supermatrix, the block Aii (=1,2,...,k) is called islated block, and the set of the elements corresponding to the block Aii is called the ith islated subsystem.

According to the irreducible standard forms of supermatrix, the systems with feedback can be classificated into four kinds:
(1). Primitive System , of which supermatrix is primitive;
(2). Circular System , of which supermatrix is not primitive but irreducible;
(3). Islated-block-primitive System, of which supermatrix is reducible and the islated blocks in its reducible standard form are all primitive;
(4). Islated-block-imprimitive System , of which supermatrix is reducible and one of islated block is imprimitive or circular.

3. The Existences and The Expressions of LIP and LAP

(1). Primitive System

The paper [1],[2] have indicated that the LIP and the LAP of primitive system exist and have given out their expressions. Here, this case is omitted.

(2). Circular System with Period C

In this case , without loss of generalization .We assume that the supermatrix of this kind of systems is taken the form of the equation (2-1). Thus we can respectively calculate $W^2, W^3, \ldots, W^c$ and W^{kc+r} (where k is a nonegative integer and r=0,1,...,C-1). In general

$$W^{kc+r} = \begin{vmatrix} B_1^k & & & 0 \\ & B_2^k & & \\ & & \ddots & \\ 0 & & & B_c^k \end{vmatrix} W^r \qquad (3-1)$$

Since B_j (j=1,2,...,c) is primitive and statistic for every j, $B'_j\ (=\lim_{k\to\infty} B_j^k)$ exists.Lonsequently $\lim_{k\to\infty} w^{kc+r}$ exist also,but its value varies with different r. So $\lim_{k\to\infty} w^k$ does not exist. In this case, the mean limit impact priority is defined [1],[2],[3] and can be expressed in matrix as follows

$$\bar{W}^{\infty}=1/c \lim_{k\to\infty}\sum_{r=0}^{c-1} W^{kc+r} \quad =1/c \lim_{k\to\infty} W^{kc} \sum_{r=0}^{c-1} W^{r} \qquad (3\text{-}2)$$

As W has a maximum characteristic value equal to 1, the matrix (I-W) is singular, and the equation (3-2) can not be simplificated as

$$\bar{W}^{\infty}=1/c\ (I-W^{c})(I-W)^{-1}(W^{c})^{\infty} \qquad (3\text{-}3)$$

Where $(W^{c})^{\infty}=\lim_{k\to\infty}(W^{c})^{k}$ and I is a unit matrix. By calculating , we obtain

$$\sum_{r=0}^{c-1} W^{r} = \begin{vmatrix} I & W_C W_{c-1} \ldots W_3 W_2 & \ldots & W_c \\ W_1 & I & \ldots & W_1 W_c \\ W_2 W_1 & W_2 & . & W_2 W_1 W_c \\ \vdots & \vdots & \ddots & \vdots \\ W_{c-1} W_{c-2} \ldots W_2 W_1 & W_{c-2} W_{c-3} \ldots W_3 W_2 & \ldots & I \end{vmatrix} \qquad (3\text{-}4)$$

$$\bar{W}^{\infty}=1/c \begin{vmatrix} B_1^{\infty} & & & 0 \\ & B_2^{\infty} & & \\ & & \ddots & \\ 0 & & & B_c^{\infty} \end{vmatrix} \sum_{r=0}^{c-1} W^{r} \qquad (3\text{-}5)$$

Let α_i stands for the main characteristic vector of B_i $(i=1,2,\ldots c)$. Notice that each column of B_i^{∞} is equal to α_i and each block of the right side of the equation (3-4) is column statistic, it follows

$$\bar{W} =1/c \begin{vmatrix} \alpha_1 & \alpha_1 & \ldots\ldots & \alpha_1 \\ \alpha_2 & \alpha_2 & \ldots\ldots & \alpha_2 \end{vmatrix} \quad . \qquad (3\text{-}6)$$

$$
\begin{vmatrix}
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\alpha_c & \alpha_c & \cdots & \alpha_c
\end{vmatrix}
$$

Consequently, the mALP can be defined and expressed in vector as[1],[2],[3] follows

$$\bar{V}^{\infty}=\bar{W}^{\infty}V^{(0)}$$

Where $V^{(0)}$ is the initial vector of priority. Obviously, $\bar{V}^{\infty}$ is equal to each column of $\bar{W}^{\infty}$, i.e. $\bar{V}^{\infty}$ is independent of $V^{(0)}$

(3). Inlated-block-primitive System

In the reducible from (2-2), if there is not final column and final row blocks, the conclusion is obvious so this case is omitted.

In general, we assume the supermatrix w is of the form (2-2), in which Aii (i=1,2,...,k) is primitive.

Theorem 1. For the system of which supermatrix is given just above.Its LIP exists and can be written in matrix as follows

$$
W^{\infty}=\begin{vmatrix}
A_{11}^{\infty} & & 0 & A_{11}^{\infty}B_1(I-Q)^{-1} \\
 & \ddots & & \vdots \\
0 & & A_{kk}^{\infty} & A_{kk}^{\infty}B_k(I-Q)^{-1} \\
 & & & \\
 & & 0 & 0
\end{vmatrix} \qquad (3\text{-}7)
$$

Where $A_{ii}^{\infty}=\lim_{p\to\infty}A_{ii}^{p}$ $(i=1,2,\ldots,k)$

Proof First the expression of W^n is showed , then let $n\to\infty$ to obtain W^{∞}.

By generalization , it follows

$$
W^{n} = \begin{vmatrix}
A_{11}^{n} & & 0 & \sum_{i=0}^{n-1} A_{11}^{i} B_1 Q^{n-i-1} \\
 & \ddots & & \vdots \\
0 & & A_{kk}^{n} & \sum_{i=0}^{n-1} A_{kk}^{i} B_k Q^{n-i-1} \\
 & & & \\
 & & 0 & Q^{n}
\end{vmatrix} \qquad (3\text{-}8)
$$

In above equation, since A_{ii} $(i=1,2,...,k)$ is primitive and statistic, and the radium of spectrum of Q is less than one , $\lim_{p\to\infty} A^p$ and $\lim_{p\to\infty} Q^p$ exist , and $\lim_{p\to\infty} Q^p=0$. Therefore we only need to consider the limit

$$\lim_{n\to\infty}\sum_{p=0}^{n-1} A_{ii}^p B_i Q^{n-p-1} .$$

Let $M=E(n/2)$, which is a integer less than $(n/2+1)$ but not less than $n/2$, $d=n-m-1$. There exits

$$\sum_{p=0}^{n-1} A_{ii}^p B_i Q^{n-p-1} = \sum_{p=0}^{m} A_{ii}^p Q^{n-p-1} + \sum_{m+1}^{n-1} A_{ii}^p B_i Q^{n-p-1} \qquad (3-9)$$
$$(i=1,2,...,k)$$

As $||A_{ii}^p||_1 =1$ and B_i is independent of P , there exists a nonegative matrix $\bar{M}$, s.t. $\bar{M} > A_{ii}^p B_i$, $\forall$ P (P is a positive integer) thus

$$\sum_{p=0}^{m} A_{ii}^p B_i Q^{n-p-1} > \bar{M}\sum_{p=0}^{m} Q^{n-p-1} = \bar{M}(I-Q^{m+1})Q^d (I-Q)^{-1} \qquad (3-10)$$

Notice $\lim_{n\to\infty} Q^d = 0$ and $\lim_{n\to\infty} Q^{m+1} =0$. So $\lim_{n\to\infty}\sum_{p=0}^{m} A_{ii}^p B_i Q^{n-p-1} =0$. On the other hand

$$\lim_{n\to\infty}\sum_{p=m+1}^{n-1} A_{ii}^p B_i Q^{n-p-1} = A_{ii}^r B_i \lim_{r\to\infty}\sum_{k=0}^{k} Q = A_{ii} B_i (I-Q)^{-1} \qquad (3-11)$$
$$(i=1,2,...,k)$$

Merging the equation (3-9) with the equation (3-10) , and calculating the limit of equation (3-8) .We obtain the equation (3-7) . The theorem 1 is proved .

Let us divide the initial priority vector $V^{(0)}$ into k+1 subvectors, such that each of them corresponds to A_{ii} $(i=1,2,...,k)$ or Q .

i.e. $V^{(0)} =(V_1^T, V_2^T ..., V_k^T, V_{k+1}^T)$, the LAP exists and can be written in vector as follows

$$V^{\infty} = W^{\infty} V^{(0)}$$

$$
= \begin{vmatrix} A_{11} V_1 \\ A_{22} V_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{kk} V_k \\ 0 \end{vmatrix} + \begin{vmatrix} A_{11} B_1 (I-Q)^{-1} \\ A_{22} B_2 (I-Q)^{-1} \\ \cdot \\ \cdot \\ \cdot \\ A_{kk} B_k (I-Q)^{-1} \\ 0 \end{vmatrix} V_{k+1} \qquad (3\text{-}12)
$$

Above equation shows that V^{∞} is independent of V^{∞} if and only if $k=1$

(4). Islated-block-imprimitive system

We omitted the system of which supermatrix is coaredient to the form (2-3) . Without loss of generalization . We assume the supermatrix is of the form (2-2) , in which one of islated blocks , e.g. A_{11} , is imprimitive , let C_i be the period of islated block A_{ii} $(i=1,2,\ldots,k)$ and C be the minimum common multiple of $C_1, C_2, \ldots, C_k$. It is indicated that the LAP of this system does not exist [1] , In this case , the mLPA $\bar{W}^{\infty}$ is defined as [1],[2],[3]

$$
\bar{W}^{\infty} = 1/c \lim_{k\to\infty} \sum_{r=0}^{c-1} W^{kc+r} = 1/c \left(\sum_{r=0}^{c-1} W^{r} \right) (W^{c})^{\infty} \qquad (3\text{-}13)
$$

Next , we simplificate above expression .

Theorem 2 . Let $\bar{A}_{ii}^{\infty} = 1/C_i \sum_{r=0}^{C_i-1} \lim_{p} A_{ii}^{r} (A_{ii}^{C_i})^{p}$, the expression (3-13) can be written as follows

$$
\bar{W} = \begin{vmatrix} \bar{A}_{ii} & & 0 & \bar{A}_{11} \left[\sum_{r=0}^{c-1} A_{11}^{r} B_1 Q^{c-r-1} \right] (I-Q^{c})^{-1} \\ & \cdot & & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \\ 0 & & \bar{A}_{kk} & \bar{A}_{kk} \left[\sum_{r=0}^{c-1} A_{kk}^{r} B_k Q^{c-r-1} \right] (I-Q^{c})^{-1} \\ & \bar{0} & & 0 \end{vmatrix}
$$

(3-14)

Proof By calculating , it is obtained

$$W^{pc} = \left| \begin{array}{cccc} A_{11}^{pc} & & 0 & \sum_{j=0}^{p-1} (A_{11}^{c})^{j} \left(\sum_{r=0}^{c-1} A_{11}^{r} B_1 Q^{c-r-1}\right)(Q^{c})^{p-j-1} \\ & \ddots & & \vdots \\ 0 & & A_{kk}^{pc} & \sum_{j=0}^{p-1} (A_{kk}^{c})^{j} \left(\sum_{r=0}^{c-1} A_{kk}^{r} B_k Q^{c-r-1}\right)(Q^{c})^{p-j-1} \\ & 0 & & Q^{pc} \end{array} \right| \tag{3-15}$$

Since there exists $\lim_{p\to\infty} A_{ii}^{pc}$ $(i=1,2,...,k)$, and $\rho(Q^{c}) < 1$.

Analagous to the proof of theorem 1 , there is the following result

$$(W^{c})^{\infty} = \left| \begin{array}{cccc} (A_{11}^{c})^{\infty} & & 0 & (A_{11}^{c})^{\infty} \left[\sum_{r=0}^{c-1} A_{11}^{r} B_1 Q^{c-r-1}\right](I-Q^{c})^{-1} \\ & \ddots & & \vdots \\ 0 & & (A_{kk}^{c})^{\infty} & (A_{kk}^{c})^{\infty} \left[\sum_{r=0}^{c-1} A_{kk}^{r} B_k Q^{c-r-1}\right](I-Q^{c})^{-1} \\ & 0 & & 0 \end{array} \right| \tag{3-16}$$

It also can be obtained

$$\sum_{r=0}^{c-1} W^{r} = \left| \begin{array}{cccc} \sum_{r=0}^{c-1} A_{11}^{r} & & 0 & \sum_{r=0}^{c-1} \left(\sum_{j=0}^{r-1} A_{11}^{j} B_1 Q^{r-j-1}\right) \\ & \ddots & & \vdots \\ 0 & & \sum_{r=0}^{c-1} A_{kk}^{r} & \sum_{r=0}^{c-1} \left(\sum_{j=0}^{r-1} A_{kk}^{j} B_k Q^{r-j-1}\right) \\ & 0 & & \sum_{r=0}^{c-1} Q^{r} \end{array} \right| \tag{3-17}$$

From the equations (3-13) , (3-16) and (3-17) , it follows

$$\bar{W} = \begin{pmatrix} \frac{1}{c}\left(\sum_{r=0}^{c-1} A_{11}^{r}\right)\left(A_{11}^{c\infty}\right) & & 0 & \frac{1}{c}\left(\sum_{r=0}^{c-1} A_{11}^{r}\right)\left(A_{11}^{c\infty}\right)\left[\sum_{r=0}^{c-1} A_{11}^{r} B_{1} Q^{c-r-1}\right]\left(I-Q^{c}\right)^{-1} \\ & \ddots & & \vdots \\ 0 & & \frac{1}{c}\left(\sum_{r=0}^{c-1} A_{kk}^{r}\right)\left(A_{kk}^{c\infty}\right) & \frac{1}{c}\left(\sum_{r=0}^{c-1} A_{kk}^{r}\right)\left(A_{kk}^{c\infty}\right)\left[\sum_{r=0}^{c-1} A_{kk}^{r} B_{k} Q^{c-r-1}\right]\left(I-Q^{c}\right)^{-1} \\ & 0 & & 0 \end{pmatrix} \quad (3-18)$$

Notice that

$$\frac{1}{c}\left(\sum_{r=0}^{c-1} A_{ii}^{r}\right)\left(A_{ii}^{c\infty}\right) = \frac{1}{c_i}\left(\sum_{r=0}^{c_i-1} A_{ii}^{r}\right)\left(A_{ii}^{c_i\infty}\right) = \bar{A}_{ii}^{\infty} \qquad (i=1,2,\ldots,k)$$

Consequently , the equation (3-18) is true and the proof is finished .

We can also illustrate the expression of mLAP like the form of the equation (3-12) .

4 . Conclusion

The computing formuli of LIP and LAP is very useful in this paper. With them we can analyse the factors affecting the priority of system with feedback , so we can steer the priority of system with feedback towards the satisfactory result more easily .

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