GROUP DECISION MAKING USING MULTIPLICATIVE AHP[1]

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Abstract: This paper discusses the methods of using the Multiplicative AHP (MAHP) for group decision making. When the number of group members is small and the views of individual members are important, the deterministic MAHP is used. Otherwise, the stochastic MAHP is employed. The literature on both these cases are summarised in this paper. In addition, methods for evaluating the consistency of a member's judgements, and evaluating the coherence of the group as a whole are derived for the case of deterministic judgements. The details are illustrated using numerical examples.

Introduction

The Analytic Hierarchy Process (AHP) (Saaty, 1980) is one of the most popular and widely employed decision tools. Its multiplicative version, called the Multiplicative AHP or MAHP, has been designed to overcome some of the criticisms of the conventional AHP (Lootsma, 1993). AHP has found a number of successful applications to group decision making (GDM) and the literature is well developed (Aczel and Saaty, 1983; Aczel and Alsina, 1986; Basak, 1988; Saaty, 1989; 1994). However, the literature on the GDM applications of MAHP shows relatively few applications. In this paper, the GDM aspects of MAHP will be discussed.

The importance of GDM either in organizational context (Desanctis and Gallupe, 1987; Huber and McDaniel, 1986) or in a larger context of social choice (Richelson, 1981) has been sufficiently stressed in the literature. The best way to reach a group decision is through unanimous consensus of all the group members. However, in practice, such unanimous consensus is not always possible, either because the members are not able to meet and discuss for sufficient period of time to enable a consensus, or because they differ in some of the basic aspects of the problem under consideration. In such cases, it is necessary to use some form of aggregation of the opinions of individual members to arrive at a compromise group opinion. AHP has been successfully applied in these situations, and is considered to be of considerable help as part of group decision support systems for promoting effective group interaction and participation (Saaty and Alexander, 1989; Saaty, 1989; 1994). Its multiplicative version also holds a similar potential. There are aggregation methods in MAHP for synthesizing group judgements from the judgements of the members. The details will be discussed in this paper.

As with the conventional AHP (See Basak and Saaty, 1993; Saaty, 1994), there are two fundamentally different appraches in MAHP for synthesising group judgements. One is deterministic, while the other is stochastic. The deterministic approach is used when a small group of individuals work closely together so that they can interact, influence and be influenced by, each other. When a large number of geographically scattered individuals are involved in making the group opinion, the stochastic approach is recommended. Further discussion on using deterministic or stochastic approach is provided in Section 4.1.

The operational issues in conducting a group decision analysis using AHP have been discussed in depth by Saaty (1989; 1994). They hold equally useful for MAHP applications. But, MAHP differs from its conventional version in the mathematical aspects of aggregation. This paper concentrates on these aspects.

[1] The research for this paper was carried out when the author visited the Faculty of Technical Mathematics and Informatics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands between September, 1995 and February, 1996.

In the case of deterministic judgements, procedures are developed in this paper for estimating the consistency of the judgements of a single decision maker, and they are extended for estimating the coherence of the group. For the case of stochastic judgements, this paper summarizes the results reported in previous studies (Honert, 1995; Ramanathan, 1996).

A brief discussion of weight derivation in MAHP is provided in the next section. Sections 3 and 4 discuss group decision making using deterministic and stochastic judgements respectively. The paper ends with a summary and conclusions.

Deriving Weights in Multiplicative AHP

Let us consider a simple hierarchical model as shown in Figure 1. The model has a goal (first level). There are a set of alternatives (last level) which have to be assigned weights in proportion to their abilities in satisfying the goal. A set of criteria (second level) are employed for evaluating the alternatives. Let there be m criteria and n alternatives. The criteria are denoted as C_j ($j = 1,2,..,m$) and the alternatives, as W_i ($i = 1,2,...,n$). We shall in this section provide an overview of how MAHP is employed for deriving weights of alternatives.

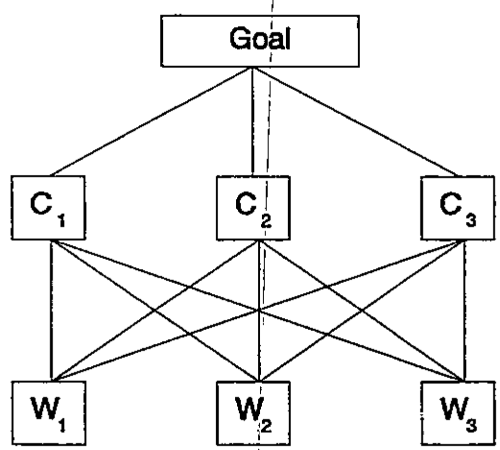

Figure 1: A Simple AHP Model

First, the process of estimating the weights of alternatives with respect to a specific criterion (say C_j) will be discussed. The alternatives are taken pairwise. The decision maker (DM) is asked to express his graded comparative judgement about a pair of alternatives W_i and W_j in terms of the relative importance of W_i over W_j with respect to the criterion under consideration. The comparative judgement is captured on a semantic scale and is converted into a numerical integer value δ_{ij} using the information in Table 1.

Table 1: Gradations of Comparative Judgements in Multiplicative AHP

Comparative Judgement	Gradation Index δ_{ij}
Very strong preference for W_i over W_j	+8
Strong preference for W_i over W_j	+6
Definite preference for W_i over W_j	+4
Weak preference for W_i over W_j	+2
Indifference between W_i and W_j	0
Weak preference for W_j over W_i	-2
Definite preference for W_j over W_i	-4
Strong preference for W_j over W_i	-6
Very strong preference for W_j over W_i	-8

Obviously, $\delta_{ii} = 0$. A justifiable condition namely $\delta_{ij} = -\delta_{ji}$ is also imposed. Intermediate integer values can be assigned to δ_{ij} in order to express a hesitation between two adjacent gradations. The DM's judgement about the pair W_i and W_j is taken to be an estimate of the preference ratio w_i/w_j, where w_i and w_j are the unknown, true weights of the two alternatives with respect to the criterion. The comparative judgements are converted into values on a geometric scale characterised by a scale parameter γ. Thus the numerical estimate of the preference ratio w_i/w_j is defined as,

$$\left(\frac{w_i}{w_j}\right) = a_{ij} = \exp(\gamma\delta_{ij}) \tag{1}$$

A value of ln $\sqrt{2}$ is recommended for γ while comparing criteria, and a value of ln 2 is recommended for comparing alternatives (Lootsma, 1993; Olson *et al.*, 1995).

Because of the condition $\delta_{ij} = -\delta_{ji}$, $n(n-1)/2$ judgements are required for estimating the weights w_i of all the alternatives. In MAHP, w_i are obtained using the logarithmic least squares technique (LLST) by minimizing the expression,

$$\sum_{i=1}^{n}\sum_{j>i}^{n}[\ln a_{ij} - (\ln w_i - \ln w_j)]^2 \tag{2}$$

subject to the multiplicative normalization condition,

$$\prod_{i=1}^{n} w_i = 1$$

It has been shown (Crawford and Williams, 1985; Lootsma, 1993; Barzilai *et al.*, 1987) that the solution for the above problem is given by the following relationship.

$$w_i = \prod_{j=1}^{n} a_{ij}^{1/n}$$

which can also be written as,

$$\ln w_i = \left(\frac{\gamma}{n}\right)\sum_{j=1}^{n}\delta_{ij} \tag{3}$$

Thus in the LLST, the weightage of the *i*th alternative is simply the geometric mean of the a_{ij} over all $j = 1,2,...,n$.

The same result will be achieved using the following mathematical programming model (Ramanathan, 1995).

$$\textit{Minimize} \sum_{i=1}^{n}\sum_{j>i}^{n} d_{ij}^2 \tag{Model 1}$$

subject to

$$\ln w_i - \ln w_j + d_{ij} = \gamma * \delta_{ij}, \quad i=1,2,..,n \;\&\; j>i$$
$$\sum_{i=1}^{n} \ln w_i = 0$$

Using a similar procedure, the weights of alternatives with respect to all the criteria can be computed, and so also the weights of criteria with respect to the goal. Let δ_{ijk} be the judgement comparing the alternative i with alternative j with respect to the criterion C_k, then the final weights of the alternative i (considering all the criteria) can be calculated using the following geometric mean aggregation procedure.

$$\ln s_i = \left(\frac{\Upsilon}{n}\right) \sum_{k=1}^{m} \sum_{j=1}^{n} c_k * \delta_{ijk}$$

where c_k is the weight of the criteria, such that $\sum_{k=1}^{m} c_k = 1$. s_i is the final weight of the alternative i, which is proportional to the ability of the alterntive i in satisfying the goal of the first level.

If the judgements are elicited from a number of persons, then the overall weight of alternative i considering the opinions of all the group members is computed using the following formula.

$$\ln f_i = \left(\frac{\Upsilon}{n}\right) \sum_{l=1}^{q} \sum_{k=1}^{m} \sum_{j=1}^{n} p_l * c_k * \delta_{ijkl} \tag{4}$$

where, f_i is the overall weight of alternative i based on the opinions of the whole group, p_l is the power (a measure of importance associated to members) of person l such that $\sum_{l=1}^{q} p_l = 1$, and q is the number of persons.

By definition, the weights (w_i, s_i and f_i etc.) are normalized multiplicatively. If needed, they can be additively normalized at the final stage.

For the remainder of this paper, we shall concentrate on the weight derivation of alternatives with respect to a single criterion, when the judgements δ_{ij} are provided by a number of persons called the *decision makers* (DM). All the members are assumed to have equal importance. In this case, (4) is modified as follows.

$$\ln g_i = \left(\frac{\Upsilon}{qn}\right) \sum_{l=1}^{q} \sum_{j=1}^{n} \delta_{ijl} \tag{5}$$

where δ_{ijl} is the judgement by DM_l for comparing alternatives i and j, and g_i is the final group weight of the alternative i. In this paper, a detailed group decision analysis using (5) is carried out. The analysis can be readily extended for the case of several criteria when dealing with deterministic judgements. However, the literature on stochastic judgements is not so well developed for dealing with several criteria.

Group Decision Making Using Deterministic Judgements

In this section, a method for evaluating the coherence of the group will be derived. As evaluation of the consistency of the judgements of each DM is necessary for evaluating coherence, this will be discussed first.

Consistency of Judgements by a Single Decision Maker

The consistency of judgements expressed by a single decision maker (DM_l) can be analyzed statistically. It is assumed that the DM_l provides $u_l = n(n-1)/2$ judgements of δ_{ijl}, which are used to obtain the weights of alternatives in his opinion (denoted as w_{il}). For example, the weight of alternative 'i' can be estimated using the following formula.

$$\ln w_{il} = \left(\frac{\Upsilon}{n}\right) \sum_{j=1}^{n} \delta_{ijl}$$

From the discussion of Section 2, the following relationship holds.

$$\frac{w_{il}}{w_{jl}} \approx a_{ijl}$$

where $a_{ijl} = \exp(\gamma\delta_{ijl})$. The approximation can be converted into equality using an error term e_{ijl} as,

$$a_{ijl} = \frac{w_{il}}{w_{jl}} e_{ijl}$$

The error terms are assumed to be independent and follow lognormal distribution. In other words, the logarithms of error terms are assumed to follow normal distribution with means zero and finite variances. Now, let the residual mean square s_l^2 be defined as follows.

$$s_l^2 = \frac{\sum_{i=1}^{n}\sum_{j>i}^{n}[\gamma\delta_{ijl} - (\ln w_{il} - \ln w_{jl})]^2}{dof_l}$$

where dof_l is the degree of freedom for DM$_l$'s judgements, which is equal to the number of independent observations minus the number of independent parameters. It is given by,

$$dof_l = u_l - (n-1) = \frac{(n-1)(n-2)}{2}$$

In this case, s_l^2 is an unbiased estimator of the variance of the error terms (Crawford and Willams, 1985). The mean μ_{er_l} of the error terms due to the DM$_l$'s judgements is given by,

$$\mu_{er_l} = \left(\frac{1}{u_l}\right) \sum_{i=1}^{n} \sum_{j>i}^{n} [\gamma\delta_{ijl} - (\ln w_{il} - \ln w_{jl})]$$

It is easy to test whether this mean is significantly different from zero (the assumed mean value of errors) using the well known hypothesis testing procedures. Let the t-statistic be defined as follows.

$$t_l = \frac{\mu_{er_l} - 0}{s_l / \sqrt{u_l}}$$

A significance level α_l is associated with the judgements of the DM$_l$ such that the value $t_{\alpha_l/2, u_l-1}$ just equals t_l. Then, α_l is a measure of consistency of the judgements of DM$_l$.

Coherence of Group Judgements

The coherence of the group, which is a measure of the consistency of the judgements of all the group members, can be obtained using a similar procedure. If all the DMs give $n(n-1)/2$ judgements each, then the total number of judgements (u) will be,

$$u = \frac{qn(n-1)}{2}$$

The residual mean square of the errors of the group judgements (s^2) is,

$$s^2 = \left(\frac{1}{u-(n-1)}\right)\sum_{l=1}^{q}\sum_{i=1}^{n}\sum_{j>i}^{n}[\gamma\delta_{ijl}-(\ln g_i-\ln g_j)]^2$$

The mean of the error terms is,

$$\mu_{er} = \left(\frac{1}{u}\right)\sum_{l=1}^{q}\sum_{i=1}^{n}\sum_{j>i}^{n}[\gamma\delta_{ijl}-(\ln g_i-\ln g_j)]$$

Finally, the t-statistic for testing the hypothesis that $\mu_{er}=0$ is given by,

$$t = \frac{\mu_{er}-0}{s/\sqrt{u}}$$

Let α be the significance level at which t just equals $t_{\alpha/2,u-1}$. Then it is a measure of the coherence of the group.

Analysis of the Results of GDM

The following terms are defined to characterise the GDM process.

The Absolute Deviation (AD_{il}) of a member l from group judgement for an alternative i can be defined as,

$$AD_{il} = |\ln g_i - \ln w_{il}|$$

The *Total Absolute Deviation* (*TAD*) of the GDM process is defined as the sum of all AD_{il}.

$$TAD = \sum_{l=1}^{q}\sum_{i=1}^{n}AD_{il}$$

The *Degree of Absolute Deviation of member l* (DAD_l) from the compromise group weights is defined as the ratio of the deviation due to all his judgements to the Total Absolute Deviation.

$$DAD_l = \frac{\sum_{i=1}^{n}AD_{il}}{TAD}$$

The *most sensitive judgement* of the GDM process is the one having the largest value of AD_{il}.

Illustration

The above procedures will be illustrated in this section for a hypothetical problem involving four alternatives and four members. The judgements are given in the matrix below. The rows and columns represent alternatives, while the values within parantheses in each cell of the matrix represent the judgements of the four members. Only the elements of the upper diagonal are shown. The other elements are automatically defined as described in Section 2. The results are tabulated in Table 2.

	j=1	2	3	4
i=1		(0,0,1,1)	(1,2,3,2)	(3,4,5,4)
2			(1,2,2,2)	(4,4,4,3)
3				(1,2,2,1)
4				

Table 2: Results of the Deterministic MAHP

Member	w_1	w_2	w_3	w_4	μ_{cr}	s^2	t	α (in %)
1	2.000	2.378	0.841	0.250	-0.0578	0.240	-0.289	78.46
2	2.828	2.828	0.707	0.177	0	0	-	100
3	4.757	2.378	0.595	0.149	0	0	-	100
4	3.364	2.000	0.595	0.250	0	0.080	0	100
Group	3.084	2.378	0.677	0.201	-0.0144	0.186	-0.164	87.12

Given the group compromise weights, the absolute deviations AD_{il} can be easily calculated. The most sensitive judgement has been found to be DM_3 on alternative 1 with a value of 0.434. The total absolute deviation is 2.769. DAD for the members (in %) are: (31.28, 15.63, 31.24, 21.85). Thus, DM_1 deviated most from the group compromise.

Group Decision Making Using Stochastic Judgements

When the number of decision makers is large, the methods described in Section 3 becomes more cumbersome. In such a case, it is advisable to use the stochastic approach of MAHP (Honert, 1995; Ramanathan, 1996), where each δ_{ijl} is considered a sample point determining the distribution of δ_{ij}, and when the distribution is assumed normal.

When to Use Stochastic MAHP?

A very crude method of determining whether to use the deterministic MAHP or stochastic MAHP is based on the number (q) of decision makers. If $q > 30$, then it is possible to have more than thirty sample points for each δ_{ij}, and hence their distribution can be considered normal due to the central limit theorem. However, the stochastic MAHP can be used even if $q < 30$ in specific situations.

Consider any decision problem that pertains to a specific group of people, *e.g.*, in the board meeting of an organization. In this case, the views of individuals have to be explicitly considered and hence the deterministic MAHP is recommended. On the other hand, consider a decision problem on a larger scale, *e.g.*, policy positions on a national or international perspective, then the most important aspect is to decide on the combined opinion, rather than analyzing the views of individual persons. In these cases, it may not be possible to get the opinions of all the relevant persons due to geographical constraints or otherwise, but it is desired to incorporate all possible variations in judgements before arriving at the group opinion. Stochastic MAHP is the recommended procedure in such cases, even if the number of persons providing the judgements is less than thirty, if the assumption that the population is normally distributed is reasonable.

Stochastic MAHP

If the judgements δ_{ijl} are considered as the sample points determining the distribution of δ_{ij}, then the mean and variance for each judgement and the covariances between any pair of judgements can be computed using the following formulae.

$$\mu_{ij} = \frac{1}{q}\sum_{l=1}^{q}\delta_{ijl}$$

$$\sigma_{ij}^2 = \frac{1}{q-1}\sum_{l=1}^{q}(\delta_{ijl}-\mu_{ij})^2$$

$$\sigma_{ij,mn}^2 = \frac{1}{q-1}\sum_{l=1}^{q}(\delta_{ijl}-\mu_{ij})(\delta_{mnl}-\mu_{mn})$$

where, μ_{ij} is the mean value of the judgement δ_{ij}, σ^2_{ij} is its variance, and $\sigma^2_{ij,mn}$ is the covariance between any pair of judgements δ_{ij} and δ_{mn}.

Interval Estimates of Weights

The mean and variance associated with the logarithms of weights can be expressed as follows (Honert, 1995).

$$\mu_{\ln w_i} = \frac{\gamma}{n}\sum_{j=1}^{n}\mu_{ij}$$

$$\sigma_{\ln w_i}^2 = \left(\frac{\gamma}{n}\right)^2\left[\sum_{ij}\sigma_{ij}^2 + \sum_{ij\neq m,n}\sigma_{ij,mn}^2\right]$$

Then, a range for w_i can be obtained as,

$$w_i = \exp\left[\mu_{\ln w_i} \pm Z_{\alpha/2}\sqrt{\sigma_{\ln w_i}^2}\right]$$

with 100(1-α) % certainty.

Point Estimates of Weights

The point estimates are arrived using the mathematical programming structure shown in Model (1). The maximum likelihood estimates of weights can be obtained by solving mathematical programming problem (Ramanathan, 1996).

$$\textit{Minimize}\ \ D'\Sigma^{-1}D \qquad \text{(Model 2)}$$

subject to

$$\ln w_i - \ln w_j + d_{ij} = \gamma * \delta_{ij}, \quad i=1,2,..,n \ \&\ j>i$$

$$\sum_{i=1}^{n}\ln w_i = 0$$

where Σ is the variance-covariance matrix associated with the judgements. D is the column vector of all the deviations (d_{ij}) as defined in the constraints of Model (1). Note that it is a *column vector* with size [$n(n-1)/2$ x 1]. For example, for the case of three alternatives, D is given by,

$$D = (d_{12}, d_{13}, d_{23})'$$

Assuming that the judgements are independent, the maximum likelihood estimates of weights w_i can be obtained by solving the following simplified mathematical programming model (Ramanathan, 1996).

$$\text{Minimize} \quad \sum_{i=1}^{n}\sum_{j>i}^{n}\left(\frac{d_{ij}^2}{\gamma^2 * \sigma_{ij}^2}\right) \qquad \text{(Model 3)}$$

subject to

$$\ln w_i - \ln w_j + d_{ij} = \gamma * \delta_{ij} \quad i=1,2,..,n \;\&\; j>i$$
$$\sum_{i=1}^{n} \ln w_i = 0$$

In general, the maximum likelihood weights will be different from the weights obtained employing the mean values, (*i.e.*) using μ_{ij} in (3) or using δ_{ijl} in (5). It has been shown in Ramanathan (1996) that they will be the same only when the variances of all the judgements are equal.

The confidence level β associated with the judgements is defined as the value of α for which the objective function of Model (3) equals the value of the α-fractile of the chi-square distribution with $n(n-1)/2$ degrees of freedom.

Illustration

The above procedures will be illustrated for the example described in Section 3.4. Even though the number of decision makers is small, the assumption of normally distributed judgements is made for the purpose of this illustration.

The mean and variance associated with the judgements are given in the matrix below. The first value in parentheses represents the mean value of δ_{ij}, while the second value represents the corresponding variance.

	j=1	2	3	4
i=1		(0.5,0.33)	(2,0.67)	(4,0.67)
2			(1.75,0.25)	(3.75,0.25)
3				(1.5,0.33)
4				

The interval estimates are given in Table 3. Notice that the ranges of w_1 and w_2 overlap, and hence, there is a probability of rank-reversal between the two alternatives. As the ranges are functions of $Z_{\alpha/2}$, it is possible to find the value of α, which just avoids the rank reversal. Such a value is found to be α=0.7517. Hence, the confidence level associated with no rank reversal can be estimated to be, 2*(1-0.7517) or 49%.

Table 3: Interval Estimates of Weights for the Stochastic MAHP

Alternative	$\mu_{\ln wi}$	$\exp(\mu_{\ln wi})$	$\sigma^2_{\ln wi}$	Interval Estimate
1	1.126	3.084	0.050	(4.458, 2.134)
2	0.866	2.378	0.025	(3.083, 1.835)
3	-0.390	0.677	0.038	(0.931, 0.492)
4	-1.603	0.201	0.038	(0.276, 0.146)

Note that the weights obtained using mean values *i.e.*, using μ_{ij} in (3) (see column 3 of Table 3) are the same as the weights obtained using deterministic MAHP *i.e.*, using δ_{ijl} in (5) (see Table 2).

The maximum likelihood weights are: w_i = (3.142, 2.389, 0.666, 0.200). The objective function value (of Model 3) is 2.635. The chi-square value (for 6 degrees of freedom) just equals the objective function value at $\alpha = 85\%$. This means that the confidence (β) that can be associated with the optimality of the solution is 85%.

It has to be noted that the deviations depend upon both, the consistency *as well as* the stochasticity of the judgements, and hence the confidence level of the maximum likelihood weights thus obtained is a measure of both consistency and stochasticity of the judgements. It is possible to approximately estimate the effect of consistency alone using the procedure outlined in Section 3.1 by employing the mean values of stochastic judgements. It is estimated to be 83.14%. Thus, it is recommended that the confidence level of the maximum likelihood estimates be viewed along with the confidence level related to the consistency of the stochastic judgements.

Unlike the case of deterministic judgements, it is not possible to extend the above procedures for dealing with several criteria. The literature on stochastic MAHP do not yet have a methodology to treat both, the stochasticity in criteria judgements *and* stochasticity in the judgements of alternatives in terms of individual criteria, simultaneously. This forms an area for further research.

Summary and Conclusions

The process of weight derivation using Multiplicative AHP has been briefly reviewed in this paper. It has been shown that group decision making can be analyzed using either deterministic MAHP or stochastic MAHP. Methods for evaluating the consistency of the judgements provided by a single decision maker, and for evaluating the coherence of the group have been developed in this paper while dealing with deterministic judgements. The methods available in the MAHP literature for dealing with stochastic judgements have been summarised. All the methods have been illustrated using numerical examples.

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