AN INCOMPLETE DESIGN IN THE ANALYTIC HIERARCHY PROCESS

Yujin Shen
The Department of Mathematics
The Stockton State College, Pomona, NJ, 08240, U.S.A.

A. E. Hoerl
The Department of Mathematical Sciences
The University of Delaware, Newark, DE, 19716, U.S.A.

Wes McConnell
The Scott Paper Company
Scott Plaza III, Philadelphia, PA, 19113, U.S.A.

ABSTRACT

In the 1970's, Thomas Saaty (1977) developed the Analytic Hierarchy Process which is a decision analysis tool. For each level of the hierarchy, pairwise comparisons are made by each judge. However, the major drawback for the method of paired comparisons is the large number of pairwise comparisons involved. In general, there have been a lot of papers dealing with reducing the number of pairs to be compaired since L. L. Thurstone developed paired comparison scales 60 years ago. But as far as AHP is concerned, there are only other three authers, E. N. Weiss and V. R. Rao, and P. T. Harker tried to deal with incomplete design for the AHP. Our approach is different from theirs. We divide the objects of a level into several subsets such that all theses subsets have one common objects as a standard one. Then pairwise comparisons are performed for each subset and a weight vector is found by solving the corresponding eigenvector problem. Finally, a weight vector for the objects of the level of the hierarchy is derived by using the common object and weight vector of each subset. Our incomplete design is based on an important property of the comparison matrix, which is Theorem 1. A concrete example, which is given by Saaty(1977), from the wealth of nations is chosen to compare the results of the incomplete design with the results of the complete design and the two results are surprisingly close to each other.

1. Introduction

In the 1970's, Thomas Saaty (1977) developed the Analytic Hierarchy Process (AHP) which is a decision analysis tool. For each level of the hierarchy, $\frac{n(n-1)}{2}$ pairwise comparisons are made by each judge where n is the number of objects in a level of the hierarchy. Then a pairwise comparison matrix $\hat{A}$ is formed. The estimated weight vector $\hat{w}$ is found by solving the following eigenvector problem:

$$\hat{A}\hat{w} = \lambda_{max}\hat{w}$$

where λ_{max} is the principal eigenvalue of $\hat{A}$. However when n is large, the number of pairs to be compared is very large. Hence an incomplete or fractional design, which will reduce the number of pairs to be compared, is necessary to handle this problem. Since L. L. Thurstone developed paired comparisons scale 60 years ago, there have been a lot of papers dealing with reducing the number of pairs to be compaired. Interested reades are referred to H. A. David(1969) and W. S. Torgerson(1958). But as far as AHP is concerned, there are only other three authors, Weiss and Rao (1984), and Harker (1986, 1987), tried to deal with incomplete design for the Analytic Hierarchy Process. Essentially, their approaches use the existing entries of the comparison matrix $\hat{A}$ to estimate the missing entries of this matrix $\hat{A}$. Then based on this completed matrix $\hat{A}$, the following eigenvector problem:

$$\hat{A}\hat{w} = \lambda_{max}\hat{w}$$

is solved and hence $\hat{w}$ is found. In the approach of Weiss and Rao, they basically apply the method of balanced incomplete design to AHP. However to apply the balanced incomplete design to AHP, the following two conditions must be satisfied:

$$bk = rt$$

$$k(r-1) = c(t-1)$$

where t = number of objects in the level; k = number of objects in the subset; b = number of subsets; r = number of replications of any objects in one administration of the b subsets; and c = times a pair of objects is replicated in one administration. Instead of estimating the missing entries in the comparison matrix $\hat{A}$, Harker (1987) sets the missing entry a_{ij} of $\hat{A}$ equal to $\frac{w_i}{w_j}$ and hopes to derive the necessary theory for his incomplete design. Unfortunately his Theorem 4 is false and the proof of his Theorem 4 is wrong. Therefore, his method for incomplete design based on his Theorem 4 lacks theoretical foundation. See Shen (1987). Our approach is different from theirs. We do not estimate the missing entries of the comparisons matrix $\hat{A}$ by its existing entries. We arbitrarily divide the objects of a level into several subsets such that all these subsets have one common object as a standard one. Then pairwise comparisons are performed for each subset and a weight vector is found. Finally, a weight vector for the objects of the level of the hierarchy is derived by using the common object and weight vector of each subset. Moreover, we are going to develop necessary theories, which are Theorem 1 and 2, to show why this incomplete design works. Dividing the objects into several subsets is not a new idea. W. S. Torgerson(1958) mentioned the similar idea before. But nobody has ever tried to apply this idea to AHP and to show why this idea works before.

2. Theory and example

Let us first state a theorem which is the base for the fractional design.

Theorem 1. Suppose we have n objects, $O_1, \ldots, O_n$, and their weights $w_1, \ldots, w_n$. Let

$$A = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_{s_1}} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_{s_1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_{s_1}}{w_1} & \frac{w_{s_1}}{w_2} & \cdots & \frac{w_{s_1}}{w_{s_1}} \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} \frac{w_{i_1+1}}{w_{i_1+1}} & \frac{w_{i_1+1}}{w_{i_1+2}} & \cdots & \frac{w_{i_1+1}}{w_{i_2}} \\ \frac{w_{i_1+2}}{w_{i_1+1}} & \frac{w_{i_1+2}}{w_{i_1+2}} & \cdots & \frac{w_{i_1+2}}{w_{i_2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_{i_2}}{w_{i_1+1}} & \frac{w_{i_2}}{w_{i_1+2}} & \cdots & \frac{w_{i_2}}{w_{i_2}} \end{pmatrix},$$

$$\cdots$$

$$A_{kk} = \begin{pmatrix} \frac{w_{i_{k-1}+1}}{w_{i_{k-1}+1}} & \frac{w_{i_{k-1}+1}}{w_{i_{k-1}+2}} & \cdots & \frac{w_{i_{k-1}+1}}{w_n} \\ \frac{w_{i_{k-1}+2}}{w_{i_{k-1}+1}} & \frac{w_{i_{k-1}+2}}{w_{i_{k-1}+2}} & \cdots & \frac{w_{i_{k-1}+2}}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_{i_{k-1}+1}} & \frac{w_n}{w_{i_{k-1}+2}} & \cdots & \frac{w_n}{w_n} \end{pmatrix},$$

where $1 < i_1 < i_2 < \cdots < i_{k-1} < n$. Let

$$\vec{w}_1^t = (w_1, \cdots, w_{i_1}),$$

$$\vec{w}_2^t = (w_{i_1+1}, \cdots, w_{i_2}),$$

$$\cdots$$

$$\vec{w}_k^t = (w_{i_{k-1}+1}, \cdots, w_n),$$

$$\vec{w}^t = (w_1, \cdots, w_n).$$

Then we have

$$A\vec{w} = n\vec{w},$$

$$A_{11}\vec{w}_1 = i_1\vec{w}_1,$$

$$A_{22}\vec{w}_2 = (i_2 - i_1)\vec{w}_2,$$

$$\cdots$$

$$A_{kk}\vec{w}_k = (n - i_{k-1})\vec{w}_k.$$

The proof of Theorem 1 is very straightforward. We can verify the above equalities by matrix multiplication.

What does Theorem 1 imply? If the weights w_1, ..., w_n are given, Theorem 1 implies that we can do pairwise comparisons among objects $O_1, \cdots, O_{i_1}$, do pairwise comparisons among objects $O_{i_1+1}, \cdots, O_{i_2}$, $\cdots$, and finally do pairwise comparisons among objects $O_{i_{k-1}+1}, \cdots, O_n$. Then we derive their weight vectors $\vec{w}_1, \cdots, \vec{w}_k$, respectively. The vector

$$\begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vdots \\ \vec{w}_k \end{pmatrix}$$

is just the weight vector $\vec{w}$ through the complete pairwise comparisons among objects $O_1, O_2, \cdots, O_n$.

But in reality, we do not know what the weight vectors $\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k$ and $\vec{w}$ are. How can we apply Theorem 1 in practice to find a suitable estimate for $\vec{w}$?

An intuitive approach would be to use Saaty's method to estimate the smaller weight vectors, $\vec{w}_i$'s, as we did in the complete design and then try to get the estimated weight vector for $\vec{w}$ from these estimated weight vectors $\hat{\vec{w}}_1, \hat{\vec{w}}_2, \cdots, \hat{\vec{w}}_k$.

Let us first look at the following example. The example will give us an idea on how to answer the above question. Suppose we have 14 objects $O_1, O_2, \cdots, O_{14}$. We then divide the 14 objects into two groups as follows:

Group 1: $O_1, O_2, \cdots, O_7$,
Group 2: $O_8, O_9, \cdots, O_{14}$.

Then we do pairwise comparisons within each group. We get the following two comparison matrices $B = (b_{ij})$ and $D = (d_{ij})$. They both are 7×7 matrices. Find the $\lambda_{max}^{(B)}$ for the matrix B and $\lambda_{max}^{(D)}$ for the matrix D. Also, we find the normalized eigenvector

$$\hat{\vec{w}}_1^t = (\hat{w}_1^{(1)}, \hat{w}_2^{(1)}, \cdots, \hat{w}_7^{(1)})$$

corresponding to $\lambda_{max}^{(B)}$ and the normalized eigenvector

$$\hat{\vec{w}}_2^t = (\hat{w}_1^{(2)}, \hat{w}_2^{(2)}, \cdots, \hat{w}_7^{(2)})$$

corresponding to $\lambda_{max}^{(D)}$. The vectors $\hat{\vec{w}}_1'$ and $\hat{\vec{w}}_2$ are the estimated weight vectors for Group 1 and Group 2, respectively. The question arises: How can we combine these $\hat{\vec{w}}_1$ and $\hat{\vec{w}}_2$ together to get an estimated weight vector for the 14 objects $O_1, O_2, \cdots, O_{14}$? If you just simply put these $\hat{\vec{w}}_1$ and $\hat{\vec{w}}_2$ together and let

$$\hat{\vec{w}}^t = (\hat{w}_1^{(1)}, \hat{w}_2^{(1)}, \cdots, \hat{w}_7^{(1)}, \hat{w}_1^{(2)}, \hat{w}_2^{(2)}, \cdots, \hat{w}_7^{(2)})$$

be an estimated weight vector for the 14 objects $O_1, O_2, \cdots, O_{14}$, then it is meaningless, because $\hat{\vec{w}}_1$ and $\hat{\vec{w}}_2$ are all relative weight vectors within each group.

Now, we divide the 14 objects $O_1, O_2, \cdots, O_{14}$ into two groups such that the two groups have one common object, say, O_1 as follows:

Group1: $O_1, O_2, O_3, O_4, O_5, O_6, O_7$
Group2: $O_1, O_8, O_9, O_{10}, O_{11}, O_{12}, O_{13}, O_{14}$.

In other words, we take O_1 as our benchmark for the two groups. After we finish pairwise comparisons for each group, we get the two comparison matrices: $B = (b_{ij})$ for Group 1 and $D = (d_{ij})$ for Group 2 where B is a 7×7 matrix and D is an 8×8 matrix now. In the same way as before, we get two estimated weight vectors $\hat{\vec{w}}_1^t = (\hat{w}_1^{(1)}, \hat{w}_2^{(1)}, \cdots, \hat{w}_7^{(1)})$ for Group 1 and $\hat{\vec{w}}_2^t = (\hat{w}_1^{(2)}, \hat{w}_2^{(2)}, \cdots, \hat{w}_8^{(2)})$ for Group 2.

Notice that the vectors are ratio scales unique up to a constant multiple. And also notice that $\hat{w}_1^{(1)}$ and $\hat{w}_1^{(2)}$ are the estimated weights of O_1 in Group 1 and Group 2, respectively. Thus, if we divide each component of $\hat{\vec{w}}_1$ by $\hat{w}_1^{(1)}$ and divide each component of $\hat{\vec{w}}_2$ by $\hat{w}_1^{(2)}$, then we get the following two new estimated weight vectors:

$$(1, \frac{\hat{w}_2^{(1)}}{\hat{w}_1^{(1)}}, \frac{\hat{w}_3^{(1)}}{\hat{w}_1^{(1)}}, \cdots, \frac{\hat{w}_7^{(1)}}{\hat{w}_1^{(1)}}),$$

$$(1, \frac{\hat{w}_2^{(2)}}{\hat{w}_1^{(2)}}, \frac{\hat{w}_3^{(2)}}{\hat{w}_1^{(2)}}, \cdots, \frac{\hat{w}_8^{(2)}}{\hat{w}_1^{(2)}}).$$

Now, each component in the above two vectors is the estimated weight relative to the estimated weight of object O_1. Thus, we simply put the above two vectors together as follows:

$$(1, \frac{\hat{w}_2^{(1)}}{\hat{w}_1^{(1)}}, \frac{\hat{w}_3^{(1)}}{\hat{w}_1^{(1)}}, \cdots, \frac{\hat{w}_7^{(1)}}{\hat{w}_1^{(1)}}, \frac{\hat{w}_2^{(2)}}{\hat{w}_1^{(2)}}, \frac{\hat{w}_3^{(2)}}{\hat{w}_1^{(2)}}, \cdots, \frac{\hat{w}_8^{(2)}}{\hat{w}_1^{(2)}})$$

and normalize to 1. We thus get an estimated weight vector for the 14 objects $O_1, O_2, \cdots, O_{14}$.

Let us give a concrete example to validate the above approach before we give a mathematical proof for this approach. The following example is taken from Thomas L. Saaty(1977).

Consider the problem of measuring the world influence of nations. We assume that influence is a function of several factors. We only consider the single factor of wealth. Seven countries were selected for this analysis. They are the United States, the U.S.S.R., China, France, the United Kingdom, Japan, and West Germany. The question is: How much more strongly does one nation as compared with another contribute its wealth to gain world influence?

Table 1

Comparative Wealth Contributions of Nations

$$\hat{A} = \begin{array}{c} \\ U.S. \\ USSR \\ China \\ France \\ U.K \\ Japan \\ W.Ger. \end{array} \begin{array}{c} \begin{array}{ccccccc} U.S. & USSR & China & France & U.K. & Japan & W.Ger. \end{array} \\ \begin{pmatrix} 1 & 4 & 9 & 6 & 6 & 5 & 5 \\ \frac{1}{4} & 1 & 7 & 5 & 5 & 3 & 4 \\ \frac{1}{9} & \frac{1}{7} & 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{7} & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{5} & 5 & 1 & 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{5} & 5 & 1 & 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{3} & 7 & 3 & 3 & 1 & 2 \\ \frac{1}{5} & \frac{1}{4} & 5 & 3 & 3 & \frac{1}{2} & 1 \end{pmatrix} \end{array}.$$

Table 1 gives the pairwise comparison matrix of the wealth contributed by the seven countries from Saaty.

The largest eigenvalue of the above matrix is 7.61 and its normalized eigenvector is

$$(.429, .231, .021, .053, .053, .119, .095).$$

The vector of ratios of the actual GNP's to the combined GNP's for the seven countries is

$$(.413, .225, .043, .069, .055, .104, .091).$$

Now let us divide the seven countries into two groups such that the U.S. is a benchmark as follows: Group 1: U.S., USSR, China, France, and Group 2: U.S., U.K., Japan, W. Germany. For Group 1, we have the following comparison matrix

$$\hat{A}_{11} = \begin{array}{c} \\ U.S. \\ USSR \\ China \\ France \end{array} \begin{array}{c} \begin{array}{cccc} U.S. & USSR & China & France \end{array} \\ \begin{pmatrix} 1 & 4 & 9 & 6 \\ \frac{1}{4} & 1 & 7 & 5 \\ \frac{1}{9} & \frac{1}{7} & 1 & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{5} & 5 & 1 \end{pmatrix} \end{array}.$$

The largest eigenvalue of $\hat{A}_{11}$ is 4.3742 and its eigenvector is (.8994, .4064, .0558, .1512). For Group 2, we have the following comparison matrix $\hat{A}_{22}$

$$\hat{A}_{22} = \begin{array}{l} \\ U.S. \\ U.K. \\ Japan \\ W.Germany \end{array} \begin{array}{c} \begin{array}{cccc} U.S. & U.K. & Japan & W.Germany \end{array} \\ \begin{pmatrix} 1 & 6 & 5 & 5 \\ \frac{1}{6} & 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & 3 & 1 & 2 \\ \frac{1}{5} & 3 & \frac{1}{2} & 1 \end{pmatrix} \end{array}.$$

The larges eigenvalue of $\hat{A}_{22}$ is 4.1674 and its eigenvector is

$$(.9553, .0991, .2770, .1966).$$

We take .8994 to divide the eigenvector:

$$(.8994, .4064, .0558, .1512)$$

and take .9553 to divide the eigenvector:

$$(.9553, .0991, .2770, .1966).$$

Then we get the following two eigenvectors:

Group 1: (1, .4519, .0620, .1681),
Group 2: (1, .1060, .2962, .2102).

Now each component in the above two vectors is the weight based on the weight of the U.S.. Therefore, we can combine these two vectors together to get the following vector:

$$(1, .4519, .0620, .1681, .1060, .2962, .2102).$$

We make the sum of components of the vector above to be one. We then get the estimated weights for the seven countries:

$$(.4358, .1970, .0270, .0733, .0462, .1291, .0916)$$

which is a good estimator.

Table 2
Weight Vectors for the Wealth Problem

	Complete Design	Fractional Design	GNP 1972	Fraction of GNP
U.S.	.429	.4358	1167	.413
USSR	.231	.1970	635	.225
China	.021	.0270	120	.043
France	.053	.0733	196	.069
U.K.	.053	.0462	154	.055
Japan	.119	.1291	294	.104
W.Ger.	.095	.0916	257	.091

Compare the weight vectors derived from the complete design and the fractional design with the actual GNP fractions given in the last column in the Table 2. They are very close in their values. (Estimates of the actual GNP of China range from 74 billion to 128 billion dollars at that time.)

Why does the above approach work? We explain mathematically. Suppose $a_1, a_2, a_3, a_4, a_5, a_6$, and a_7 are the true GNP's for each of the seven countries, the U.S., the USSR, China, France, the U.K., Japan, and W. Germany. We divide them into two groups: Group 1: a_1, a_2, a_3, a_4 for U.S., USSR, China and France; Group 2: a_1, a_5, a_6, a_7 for U.S., U.K., Japan and W.Germany. Let

$$T_1 = a_1 + a_2 + a_3 + a_4,$$

$$T_2 = a_1 + a_5 + a_6 + a_7.$$

Then $\frac{a_1}{T_1}$, $\frac{a_2}{T_1}$, $\frac{a_3}{T_1}$ and $\frac{a_4}{T_1}$ are the fractions of GNP total for the four countries of Group 1, and $\frac{a_1}{T_2}$, $\frac{a_5}{T_2}$, $\frac{a_6}{T_2}$ and $\frac{a_7}{T_2}$ are fractions of GNP total for the four countries of Group 2. Now we take $\frac{a_1}{T_1}$ to divide the vector $(\frac{a_1}{T_1}, \frac{a_2}{T_1}, \frac{a_3}{T_1}, \frac{a_4}{T_1})$ and take $\frac{a_1}{T_2}$ to divide the vector $(\frac{a_1}{T_2}, \frac{a_5}{T_2}, \frac{a_6}{T_2}, \frac{a_7}{T_2})$, we get the following two vectors:$(1, \frac{a_2}{a_1}, \frac{a_3}{a_1}, \frac{a_4}{a_1})$ and $(1, \frac{a_5}{a_1}, \frac{a_6}{a_1}, \frac{a_7}{a_1})$. In fact, the vectors (1, .4519, .0620, .1681) and (1, .1060, .2960, .2012) are the estimates of the above two vectors, respectively.

Now we drop the 1 in $(1, \frac{a_5}{a_1}, \frac{a_6}{a_1}, \frac{a_7}{a_1})$ and concatenate the two vectors, $(1, \frac{a_2}{a_1}, \frac{a_3}{a_1}, \frac{a_4}{a_1})$ and $(\frac{a_5}{a_1}, \frac{a_6}{a_1}, \frac{a_7}{a_1})$. Thus we get the following vector:

$$(1, \frac{a_2}{a_1}, \frac{a_3}{a_1}, \frac{a_4}{a_1}, \frac{a_5}{a_1}, \frac{a_6}{a_1}, \frac{a_7}{a_1}).$$

Let

$$\begin{aligned} T_3 &= 1 + \frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_4}{a_1} + \frac{a_5}{a_1} + \frac{a_6}{a_1} + \frac{a_7}{a_1} \\ &= \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7}{a_1} \end{aligned}$$

and

$$\begin{aligned} T &= a_1 T_3 \\ &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7. \end{aligned}$$

Then we take T_3 to divide the vector:

$$(1, \frac{a_2}{a_1}, \frac{a_3}{a_1}, \frac{a_4}{a_1}, \frac{a_5}{a_1}, \frac{a_6}{a_1}, \frac{a_7}{a_1}).$$

We get the following vector:

$$(\frac{a_1}{T}, \frac{a_2}{T}, \frac{a_3}{T}, \frac{a_4}{T}, \frac{a_5}{T}, \frac{a_6}{T}, \frac{a_7}{T}),$$

which are the fractions of the GNP total for the seven countries.

The above argument indicates why the fractional design approach which we have applied to get the estimated weight vector for the seven countries is reasonable.

The above approach can be generalized into the following theorem.

Theorem 2. Suppose we have k groups of objects such that each group has one common object.

Group 1: $O_{11}, \cdots, O_{1n_1}$,
Group 2: $O_{21}, \cdots, O_{2n_2}$,
$\cdots$
Group k: $O_{k1}, \cdots, O_{kn_k}$.

Assume $O_{11}, O_{21}, \cdots, O_{k1}$ are all the same objects. Also assume, for each group, there exists the true normalized weight vector $(w_{i1}, w_{i2}, \cdots, w_{in_i})$ for $i = 1, 2, \cdots, k$. Let $w'_{i1} = \frac{w_{i1}}{w_{i1}}, w'_{i2} = \frac{w_{i2}}{w_{i1}}, \cdots, w'_{in_i} = \frac{w_{in_i}}{w_{i1}}$ for $i = 1, 2, \cdots, k$.

Let

$$T_1 = 1 + \sum_{i=1}^{k} \sum_{j=2}^{n_i} w'_{ij}.$$

Then the vector

$$\vec{w}' = (\frac{1}{T_1}, \frac{w'_{12}}{T_1}, \cdots, \frac{w'_{1n_1}}{T_1}, \frac{w'_{22}}{T_1}, \cdots, \frac{w'_{2n_2}}{T_1}, \cdots, \frac{w'_{k2}}{T_1}, \cdots, \frac{w_{kn_k}'}{T_1})$$

is the normalized weight vector for the whole group of objects

$$(O_{11}, \cdots, O_{1n_1}, O_{22}, \cdots, O_{2n_2}, \cdots, O_{k2}, \cdots, O_{kn_k}).$$

Proof: The proof is omitted. The idea of the proof is the same as the last example.

Of course the true normalized weight vector $(w_{i1}, w_{i2}, \cdots, w_{in_i})$ for each Group i is unknown in practice where $i = 1, 2, \cdots, k$. But we can use the normalized eigenvector corresponding to λ_{max} of the pairwise comparison matrix for Group i to estimate the true normalized weight vector $(w_{i1}, w_{i2}, \cdots, w_{in_i})$ where $i = 1, 2, \cdots, k$. Substituting w_{ij} by its estimate, $\hat{w}_{ij}$, we then can get the $\hat{\vec{w}}$ which is an estimate for $\vec{w}$, i.e., an estimate for the normalized weight vector for the whole group of objects

$$(O_{11}, \cdots, O_{1n_1}, O_{22}, \cdots, O_{2n_2}, \cdots, O_{k2}, \cdots, O_{kn_k}).$$

Now let us return to the example of the wealth contributions of nations for their world influence. We have chosen the U.S. as the common object in the fractional design we have used. Of course we do not have to choose the U.S. as the common object. We have tried each of the remaining six countries as the common object and calculated the estimated weight vectors. The results are, in general, close to the result of the complete design for this example.

3. Conclusions

We have proposed one method for reducing the number of pairwise comparison in AHP. The method can be used in the implementation of AHP when the number of objects in a level of the hierarchy is large. However two interesting research questions remain: 1. Is there any theoretical or empirical guideline for choosing the common object which would constitute the incomplete design? 2. There are many ways to set up subsets of objects. Does there exist a way of setting up subsets so that it would give us the best result? Before we know the answers, if they exist, to the above questions, to get a better result in using this incomplete design. one should always check the consistency index defined by Saaty in 1977 for each group. In general the more consistent the comparison matrices are, the more reliable the estimated weight vector for the whole group of objects will be.

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