An Approach to Synthesising Judgments

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ABSTRACT

We discuss the method of synthesizing judgments in the GHAP (the AHP in group decision making). Several criteria and conditions are developed from statistical considerations. Besed on the criteria and conditions, the applications of the GAHP are more effective.

1. Introduction

The Analytic Hierarchy Process (AHP) has been proven to be a very useful decision aid. It has many advantages which are the ease of use, the ability to handle in consistency in judgments and the technique to disect a decision into its less complex hierarchic construction etc. With the development of the AHP, it is found that the AHP is a suitable and precise procedure in group decision making. Theorical studies of the GAHP have carried out. Neves,J. (1984), who used empirical methods with the AHP, elaboraled the problems of monitoring consistency in group decision making. Aczél, J. and Alsina, C. (1986) put forward several conditions that is reasonable for synthesising judgments. But, until now we have not yet performed any experimental studies of the handling of devergence of group judgments.

In fact, Saaty's concept of in consistency for a judgment matrix is not, at least to some extent, inappropriate for the GAHP. We see an example.

Suppose that the specific gravities of three objectives O_1, O_2, and O_3 are 200, 120 and 80 unit, respectively. The judgment matrices of two individuals are

$$A_1 = \begin{vmatrix} 1 & 4 & 2 \\ 1/4 & 1 & 1/2 \\ 1/2 & 2 & 1 \end{vmatrix} \quad \text{and} \quad A_2 = \begin{vmatrix} 1 & 3 & 6 \\ 1/3 & 1 & 2 \\ 1/6 & 1/2 & 1 \end{vmatrix}$$

These two matrices are perfectly consistent. Using the geometric mean, we get the consistent synthesising matrix as follows:

$$A = \begin{vmatrix} 1 & \sqrt{12} & \sqrt{12} \\ 1/\sqrt{12} & 1 & 1 \\ 1/\sqrt{12} & 1 & 1 \end{vmatrix}$$

The right principal eigenvector of the matrix A is

$$(0.634, \ 0.183, \ 0.183)^T$$

For another two individuals, the judgment matrices are

$$B_1 = \begin{vmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{vmatrix} \quad \text{and} \quad B_2 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1/2 & 1 & 1 \end{vmatrix}$$

which are both in consistent. The synthesising matrix

$$B = \begin{vmatrix} 1 & \sqrt{2} & \sqrt{6} \\ 1/\sqrt{2} & 1 & \sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1 \end{vmatrix}$$

is also inconsistent. But, the eigenvector of the matrix B, $(0.504, 0.291, 0.205)^T$, is closer to the real priority $(0.5, 0.3, 0.2)^T$ than that of the matrix A, $(0.634, 0.183, 0.183)^T$. The above example explaine why the inconsistency in judgments has nothing to do with the difference between the principal vector of judgment matrix and the real priority.

If assume that the judgments in a group of individuals has certain probability distributions round the real values (for example, a lognormal distribution), the agreement level of the group opinion, which is quite different from the inconsistency, can bring to light the relationship between the order derived from synthesising matrixes and the real order. Thus, the judgment from a group of individuals can be regarded as random samples with a lognormal distribution in the GAHP. And the variance of the logarithm of the judgments can be taken as a measure of devergence of group judgments.

This paper intends to deal with the handling of group judgments with considering the devergence. The outline of the paper is as follows: Section 2 discusses the extreme judgments in group.
Scetion 3 treats the peculiar individuals.

Section 4 gives the condition for accepting group judgments.
Section 5 provides concluding remarks.

2. The distinguish and rejection of extreme judgments

Suppose that the ratio judgments denoted by $\xi_1, \xi_2, \ldots, \xi_K$, or ($\xi_1, \xi_2, \ldots, \xi_K$). of K individuats from a simple random sample from the lognormal porpulation , i.e. $\xi_1, \xi_2, \ldots, \xi_K$ are independent and identically distributed. If set

$$\eta = \ln \xi \tag{1}$$

then $\eta_1 = \ln \xi_1$, $\eta_2 = \ln \xi_2$, ..., $\eta_K = \ln \xi_K$ from a random sample from the normal porpulation with the mean μ and the variance σ^2 .

According to the statistic inference, we know that sample mean

$$\bar{\eta} = \frac{1}{K} \sum_{k=1}^{K} \eta_k \tag{2}$$

and sample variance

$$S_0^2 = \frac{1}{K-1} \sum_{k-1}^{K} (\eta_k - \bar{\eta})^2 \tag{3}$$

are the unbiased estimation of μ and σ^2 , repectively.

S_0^2 provieds a measure of the spread or dispersion of the group judgments around its mean $\bar{\eta}$. A large value of S_0^2 either typically indicates that the group judgments have a wide spread around $\bar{\eta}$ or suggests that the difference of certain an individual judgment from $\bar{\eta}$ is much larger than those of the others. If the difference is so large that it quite influences the quality of synthesising judgments, we call it the extreme judgment and have to reject it. Three criteria, which are used to distinguish and reject the extremejudgments, are presented as follows:

Criterion 1.

Let

$$V_k = \eta_k - \bar{\eta} \qquad k = 1, 2, \ldots, K \tag{4}$$

If

$$| V_k | > \beta S_0 \tag{5}$$

the judgment η_k (ξ_k) should be rejected (where β is a constant and we can take $\beta = 2$, or 3).

The Criterion 1 combines the advantages of being easy to use and to accept. But, when K is small, it fails to reject the extreme judgments. See the following example:

Suppose that a group of judgments are (1/3, 3, 4, 5, 5). Here, $K = 5$, $\bar{\eta} = 0.921$, $S_0 = 1.148$ and $|V_1| = 2.020 < 2S_0 = 2.296$. According to the Cri. 1, it is unnecessary to reject η_1 ($\xi_1 = 1/3$). But we see that there exists larger difference between the judgment 1/3 and the others. Therefore it is not suitable to keep the judgment 1/3 in the group. In order to effectively reject the extreme judgments for smaller K, Criterion 2 may be practical.

Criterion 2.

Suppose that in K judgments the probability of the events which are impossible to appear is 1/2K . If we take the judgment errors as random variances from a standard normal distribution. Then

$$1 - \frac{1}{\sqrt{2\pi}} \int_{-W_K}^{W_K} \exp(-x^2/2)\,dx = \frac{1}{2K} \qquad (6)$$

where W_K and $-W_K$ are called the critical parameters.

Noting the defination of a standard normal distribution and the round-off error 0.5, we can obtain

$$\phi (W_K) = 1/2 (1 - \frac{1}{2k}) + 0.5 = 1 - \frac{1}{4k} \qquad (7)$$

Thus, for different K, we can determine W_K from (7), as below:

K	w_K	k	w_K	k	w_K	k_K	w_K
3	1.38	10	1.96	17	2.17	24	2.31
4	1.53	11	2.00	18	2.20	25	2.33
5	1.65	12	2.03	19	2.22	30	2.39
6	1.73	13	2.07	20	2.24	40	2.49
7	1.80	14	2.10	21	2.26	50	2.58
8	1.86	15	2.13	22	2.28	70	2.71
9	1.92	16	2.15	23	2.30	100	2.81

If

$$| V_k | > w_k S_0, \qquad k = 1, 2, \ldots, K \qquad (8)$$

then reject the judgment $\eta_k(\xi_k)$.

Reconsidering the example above. we have $|v_1| = 2.020 > w_{50} S = 1.894$ with using Cri.2. S_0, η_1 can be rejected. Cri.2 complement the deficiency of Cri.1. In return for this, Cri.2 is less effective than Cri.1 when K is larger.

Another property which Cri.2 has is that it provides different confidence level for different K. If we desire to deal with problem under the same confidence level, the following Cri.3 may be needed .

Criterion 3.

Suppose that the order, from small to large, of judgments is

$$\eta_1 \leq \eta_2 \leq \cdots \leq \eta_K$$

and that η_K should probably be removed from K judgments. We introduce two new functions $\bar{\eta}_K$ and S_K^2 by

$$\bar{\eta}_K = \frac{1}{K-1} \sum_{k=1}^{K-1} \eta_k \tag{9}$$

and

$$S_K^2 = \frac{1}{K-1} \sum_{k=1}^{K-1} (\eta_k - \bar{\eta}_k)^2 \tag{10}$$

From (9) and (10), we get

$$\frac{S_K^2}{S_0^2} = 1 - \frac{K}{(K-1)^2} \left(\frac{\eta_K - \bar{\eta}}{S_0} \right)^2 \tag{11}$$

Here, both S_0^2 and S^2 are the functions of judgments: Therefore the probability density function of S_K^2 / S_0^2 can be obtained, and, furthermore, so does that of $\left(\frac{\eta_K - \bar{\eta}}{S_0} \right)^2$. Assume the probability density function of $\left(\frac{\eta_K - \bar{\eta}}{S_0} \right)^2$ is denoted by $p(\eta)$, then

$$P_r \left\{ \frac{\eta_K - \bar{\eta}}{S_0} < \lambda(\alpha, k) \right\} = 1 - \alpha \tag{12}$$

where $\lambda(\alpha, K)$ is the upper $100\alpha\%$ point of $p(\eta)$.
For given level of significance α, if

$$\eta_K - \bar{\eta} = |v_K| > \lambda(\alpha, k) S_0 \tag{13}$$

then $\eta_K(\xi_K)$ should be rejected.

The values of $\lambda(\alpha, k)$ are presented as follows:

K	α=0.05	α=0.10	K	α=0.05	α=0.10	K	α=0.05	α=0.10
3	1.15	1.13	12	2.29	2.11	21	2.58	2.34
4	1.46	1.37	13	2.33	2.14	22	2.60	2.36
5	1.67	1.56	14	2.37	2.17	23	2.62	2.37
6	1.82	1.68	15	2.41	2.20	24	2.64	2.39
7	1.94	1.81	16	2.44	2.23	25	2.66	2.41
8	2.03	1.88	17	2.47	2.26	30	2.74	2.47
9	2.11	1.95	18	2.50	2.28	35	2.81	2.52
10	2.18	2.01	19	2.53	2.30	40	2.87	2.55
11	2.24	2.06	20	2.56	2.32	50	2.96	2.60

In order to compare the features of the three criteria each other, we see the examples in the table 1.

From these examples, we see that Cri. 1 is practicable for large samples and Cri. 2. as well as Cri. 3. is better for small samples than Cri. 1. In addition to this, Cri. 3 provides an attempt to handling judgments under the same confidence level for different K.

If only two respects, the number of judgments and rejecting effective, are concerned with, it is suggested to

i. use Cri. 1 for K > 10;
ii. use Cri. 2 for b < K <= 10;
iii. ues Cri. 3 for K <= b.

Table. 1

Group		I(K=15)	II(K=8)	III(K=4)
Judgment		1/3,1/2,1,1,2, 2,3,3,3,3,4,4, 4,5,8	1/3,1/2,1/2,1, 1,1,2,5	1/5,1/3,1/2,4
Statistics		$\bar{\eta}$=.789 S_0=.879	$\bar{\eta}$=-.023 S_0=.865	$\bar{\eta}$=-.504 S_0=1.315
Results	Cri.1 (β=2)	$\lvert\eta_1-\bar{\eta}\rvert$=1.888 >$2S_0$=1.758 Reject η_1(1/3)	$\lvert\eta_8-\bar{\eta}\rvert$=1.632 <$2S_0$=1.792 Reserve η_8(5)	$\lvert\eta_4-\bar{\eta}\rvert$=1.890 <$2S_0$=2.629 Reserve η_4(4)
	Cri.2	1.888>$w_{15}S_0$=1.872 Reject η_1	1.632>w_8S_0=1.608 Reject η_8	1.890<w_4S_0=2.011 Reserve η_4

Cri.3 (α=0.1)	$1.888<\lambda(.1,15)S_q=1.937$ Reserve η_1	$1.632>\lambda(.1,8)S_0=1.626$ Reject η_8	$1.890>\lambda(.1,4)=1.801$ Reject η_4

3. Treatment of Peculiar Individuals

We know that each nxn completed individual judgment matrix in the AHP has n(n-1)/2 independent elements. If the n(n-1)/2 elements are regarded lognormal distributions, the random errors

$$\delta_{ij,k} = \ln a_{ij,k} - \bar{\eta}_{ij} \qquad (k = 1, 2, \ldots, K) \qquad (14)$$

have normal distributions with the mean 0 and the variance σ^2_k, where

$$\bar{\eta}_{ij} = \frac{1}{K} \sum_{k=1}^{K} a_{ij,k}$$

Let

$$\bar{\delta}_k = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \delta_{ij,k} \qquad (k = 1, 2, \ldots, K) \qquad (15)$$

and

$$s^2_k = \frac{1}{n(n-1)/2-1} \sum_{1<i<j<n} (\delta_{ij,k} - \bar{\delta}_k)^2$$
$$= \frac{2}{n(n-1)-2} \sum_{1\leq i<j\leq n} (\delta_{ij,k} - \bar{\delta}_k)^2 \qquad (16)$$

then statistic

$$w_k = \frac{(n(n-1)/2-1)s^2_k}{\sigma^2_k}$$

has a distribution with $(\frac{n(n-1)}{2} - 1)$ degree of freedom, and statistic

$$F_{p,q} = \frac{w_p/(n(n-1)-2/2)}{w_q/(n(n-1)-2/2)} = \frac{s^2_p}{s^2_q} \frac{\sigma^2_q}{\sigma^2_p}$$

has an F distribution with $\frac{n(n-1)-2}{2}$ and $\frac{n(n-1)-2}{2}$ degrees of fredom. Now, it is desired to test the following hypothesis at the level of significance α= 0.05 or α = 0.10.

$$H_0 : \sigma^2_p = \sigma^2_q ,$$

If

$$\frac{1}{F_{\alpha/2}(\frac{n(n-1)-2}{2},\frac{n(n-1)-2}{2})} < \frac{s^2_p}{s^2_q} < F_{\alpha/2}(\frac{n(n-1)-2}{2}, \frac{n(n-1)-2}{2}) \qquad (17)$$

H_0 should be accepted. if not, be rejected.

In general, when each individual has no (or less) rejected extreme judgments, the H_0 should be accepted for n $\sigma \in$ { 1, 2, ..., K }. That is, at the level of significance α,

$$\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_K^2 \qquad (18)$$

But, when certain an individul (for convenience, take it as K) has more extreme judgments rejected, than the there among the group, H_0 may not be accepted.

With the additivity of the χ^2 distributions, the statistic

$$W' = \sum_{k=1}^{K-1} \frac{(n(n-1)-2/2)S^2}{\sigma_k^2} = \frac{n(n-1)-2}{2} \sum_{k=1}^{K-1} \frac{S_k^2}{\sigma_k^2} \qquad (19)$$

has a χ^2 distribution with [(k-1) $\frac{n(n-1)-2}{2}$] degree of freedom.

Suppose $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_{K-1}^2 = \sigma^2$, the (19) goes over into

$$W' = \frac{n(n-1)-2}{2\sigma^2} \sum_{k=1}^{K-1} S_k^2$$

We have also statistic

$$F = \frac{W_K/(\frac{n(n-1)-2}{2})}{W'/((K-1)\frac{n(n-1)-2}{2})} = \frac{(K-1)W_K}{W'} = \frac{(K-1)S_K^2}{\sum_{k=1}^{K-1} S_k^2} \cdot \frac{\sigma^2}{\sigma_K^2} \qquad (20)$$

which has an F-distribution with $\frac{n(n-1)-2}{2}$ and (k-1) $\frac{n(n-1)-2}{2}$ degrees of freedom.

Let us test the following hypothesis at given level of significance α,

$$H_1 : \sigma_K^2 > \sigma^2$$

If

$$\frac{(k-1)S^2}{\sum_{k=1}^{K-1} S_k^2} > F_\alpha\left(\frac{n(n-1)-2}{2}, (k-1)\frac{n(n-1)-2}{2}\right) \qquad (21)$$

H_1 should be accepted, otherwise be rejected.

Thus, we get the Criterion A as follows:

Criterion A

At the given level of significance α=0.05 if

$$\frac{(k-1)S_k^2}{\sum_{k=1}^{K-1} S_k^2} > r_{0.05} F_{0.05}\left(\frac{n(n-1)-2}{2}, (k-1)\frac{n(n-1)-2}{2}\right)$$

the all judgments of Kth individuval should be rejected, where $r_{0.05}$ is called the "misjudgment factor" . It is proved that $r_{0.05} = 2$ is reasonable for the GAHP.

Similar to the analysis above, we can obtain another Criterion B:

At the level of significance $\alpha = 0.05$, if

$$\frac{(K-1) \sum_{1 \leq i < j \leq n} \delta_{ij,k}^2}{\sum_{k=1}^{K-1} \sum_{1 \leq i < j \leq n} \delta_{ij,k}^2} > r_{0.05} F_{0.05}\left(\frac{n(n-1)}{2}, (k-1)\frac{n(n-1)}{2}\right)$$

the all $\frac{n(n-1)}{2}$ judgments of Kth individul should be rejected.

We present an example as follows:

Judgment \ Individual	1	2	3	4	5	6	Error \ Individual	1	2	3	4	5	6
a12	2	3	2	[illegible]	3	4	δ_{12}	-0.2507	0.1549	-0.2507	-0.2507	0.1549	0.44[illegible]
a13	1	2	1	1/3	2	1	δ_{13}	-0.0479	0.6452	-0.0479	-1.1666	0.6452	0.0479
a14	3	4	4	3	3	3	δ_{14}	-0.0959	0.1918	0.1918	-0.0959	-0.0959	-0.0959
a23	1/3	1/3	1/2	1/5	1/2	1/5	δ_{23}	0.0351	0.0351	0.4406	-0.4757	0.4406	-0.4757
a24	2	1	2	2	1	1	δ_{24}	0.4621	-0.2310	0.4621	0.4621	-0.2310	-0.2310
a34	3	3	3	9	2	3	δ_{34}	-0.1155	-0.1155	-0.1155	-0.9931	-0.5210	-0.1155
$S^2 = \frac{1}{5}\sum_{1 \leq i < j \leq n} (\delta_{ij} - \bar{\delta})^2$								0.2439	0.3059	0.2986	0.7449	0.4269	0.3134
$\delta^2 = \sum_{1 \leq i < j \leq n} \delta_{ij}^2$								0.3025	0.5447	0.5[illegible]9	2.7970	0.9424	0.5007

It is easy to prove $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_5^2 = \sigma_6^2 = \sigma^2$, at the level of significance $\alpha = 0.05$. But, by means of Cri. A,

$$\frac{5S_4^2}{S_1^2 + S_2^2 + S_3^2 + S_5^2 + S_6^2} = 5.3084 > 2F_{0.05}(5, 25) = 5.2$$

so, the all $\frac{n(n-1)}{2}$ judgments of 4th individual should be rejected. The same result can be obtained with Cri. B.

4. The Condition for Accepting Group Judgments

Actually, the methods above mentioned are not always appropriate. When a group of judgments have a large spread around $\bar{\eta}$, the results of synthesising judgment do not reflect the group intention. For example, against the same problem, group I have judgments { 1/5, 1/3, 1, 3, 4, 5 } and group II { 1, 2, 2, 2, 3, 4 }. Here $\bar{\eta}_I = 0.231$, $\bar{\eta}_{II} = 0.761$, $S_{0I} = 1.356$ and $S_{0II} = 0.469$. It is easy to find that there are no any extreme judgment to be rejected from these two group judgments, by the criteria present above and that the group I has larger dissent of opinion than the group II has.

Suppose that the entire top triangular portion of all judgment matrices, kn(n-1)/2 judgments, are independent each other. If it follows from (18) that $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_K^2 = \sigma^2$. At the level of significance α, we can not determine and reject any peculiar individuals.

Thus, for different values of σ^2, the relations between σ^2 and C.R. , the random consistency index, can be obtained by the Monte Carlo trails. Table 2 gives the results.

The entries in the first two column describe the parameters of the Monte Carlo run. The frist number is the variance of $\ln a_{ij}$ and the second is the order of judgment matrix.

The Number of Trails = 3000 Table 2

σ^2/n	$\bar{\lambda}_{max}$	Var (λ_{max})	C.R. = $\frac{C.I.}{R.I.}$
0.25/5	5.318	0.03090	0.0711
0.30/5	5.382	0.04524	0.0854
0.35/5	5.453	0.06481	0.1013
0.40/5	5.523	0.08829	0.1169

0.50/5	5.666	0.14719	0.1489
0.25/7	7.559	0.03185	0.0693
0.30/7	7.674	0.04991	0.0835
0.35/7	7.797	0.07385	0.0988
0.40/7	7.922	0.09956	0.1143
0.50/7	8.174	0.15750	0.1455
0.25/9	9.828	0.03110	0.0708
0.30/9	10.007	0.04794	0.0861
0.35/9	10.189	0.06255	0.1017
0.40/9	10.373	0.08389	0.1174
0.50/9	10.745	0.15225	0.1492

We see from the table 2 that if $\sigma^2 < 0.35$, C.R. is easy to be satisfied, otherwise, not.

It is worth mentioning here that σ^2 deseribes not only the inaccuracy but also the inconsistency of judgments. The relation between σ^2 and C.R. implies that we can take the critical value of S_0^2 as the condition for accepting group judgment. Thus, we get following

Condition:

If, by proceeding Cri. A and Cri. B, $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma^2$ at the given level of significance α and σ^2 is greater than 0.4 with confidence coificiency 0.9 or 0.3 with confidence coidency 0.975, then the group judgments are said to be more divergent and should be rejected.

We can define the critical value S_c^2 of S_0^2 by

$$S_c^2 = \frac{0.4\chi^2_{0.1}(K-1)}{K-1}$$

The condition is equivalent to:

If $S_0^2 > S_c^2$, then reject all of the group judgments, otherwise accept.

Table 3 gives the values of S_c^2 for different K with confidence coifidency 0.9

Table 3

K	S_c^2	K	S_c^2	K	S_c^2	K	S_c^2
3	0.921	10	0.653	17	0.589	24	0.557
4	0.833	11	0.639	18	0.583	25	0.553
5	0.778	12	0.628	19	0.578	30	0.539
6	0.739	13	0.618	20	0.573	35	0.528
7	0.710	14	0.610	21	0.568	40	0.520
8	0.687	15	0.602	22	0.564	45	0.512
9	0.668	16	0.595	23	0.560	50	0.505

Now, reconsider the example presented at the beginning of this section. Since $S_{0I}^2 = 1.839 > 0.739$, $S_{0II}^2 = 0.220 < 0.739$, so the judgments of the group I should be rejected and group II, accepted.

5. Concluding Remarks

In order to put the GAHP to practice, we have discussed the methods of distinguishing and of determining divergent individuals.

These methods, in fact, are interrelated. We summarize the methodology presented in the previous sections as following steps.

Step 1:

Determine that whether or not there exist extreme judgments among all of the entries of K judgment matrices with the three criteria given in the section 2.

Step 2:

Put the individuals among whose judgment matrices there are more extreme judgments to the test of rejecting peculiar individuals with the two criteria given in the section 3. After the test, reject the peculiar individuals and put the remainder into the step 3.

Step 3:

Test the remainder whether to satisfy the condition for accepting group judgments or not. If not, reject the all judgments of the remainder. If the condition is satisfied the non-rejected judgments may be synthesised by the geometric mean. Since the judgments of each individual has some consistency, we can only make one of the corresponding elements (e.g. $a_{12,k}$) of the remainder judgment matrices to be tested.

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