FUZZY ANALYTICAL HIERACHY PROCESS AND ITS APPLICATION

Hu Zhaoguang
Electric Power Research Institute
Beijing, China

ABSTRACT

How to use fuzzy informationin hierarchical analysis is to be discussed in this paper. We use R-fuzzy sets instead of exact ratio. In order to reduce the calculation, the new concepts and operators are defined. We can see that the multiple expert's judges can be summarized in the R-fuzzy sets, and also it can reduce the calculation process. A numerical example is given and the summary which compairs with previous research is also made in the paper.

I. Introduction

Analytical Hierarchy Process (AHP) is growing into a big family since T.L. Saaty first published his paper and its various offsprings are bearing attractive results of applications, especially in decision making. However, there are many evaluations which could not be expressed in numerical measures but only in human natural language such as "important", "more important" etc. Such evaluations are difficult to be summarized in numerical values needed in AHP. However, they can be summarized in the form of fuzzy sets or fuzzy numbers. In order to make AHP suitable to these cases, a new method of AHP with fuzzy information is presented in this paper, i.e. fuzzy analylical hierarchical process. The main idea of fuzzy AHP is using fuzzy sets to express expert's evaluationof merits and to form the fuzzy matrix instesd of numerical matrix in AHP. The results can be obtained through the operation of fuzzy operator according to fuzzy set theory and also the results are expressed in the form of fuzzy sets. In order to summarize the multiple expert's evaluations in fuzzy AHP, we give a new method in this paper which can reduce the calculation in hierarchy.

There are six sections in the paper:

- I. Introduction.
- II. Fuzzy concepts and definitions, introducing some fuzzy concepts which appear in fuzzy AHP and their definitions.
- III. The fundamentals of fuzzy AHP. The main idea and mathematical fundamentals of fuzzy AHP are given.
- IV. Multiple experts. Introduce a new method to summarize expert's evaluations in a form of R-fuzzy sets.
- V. Application. An example of power plant sitting is given.
- VI. Conclusion. Give out the defferences between our method and previous ones.

II. Fuzzy concepts and definitions

In order to discuss our problem, we give the following concepts and their definitions.

Definition 1. Let universe R be a real field, then the fuzzy set A on R is called R-fuzzy set. The collection of all fuzzy sets on R is denoted by $F(R)$, then $\tilde{A} \in F(R)$.

Definition 2. Let $\tilde{A}$, $\tilde{B} \in F(R)$ and

$$\tilde{A} = \int \frac{\mu_A(x)}{x} \qquad \text{for } \mu_A(x) = \mu_B(y) \text{ where x according to y only one to one.}$$

$$\tilde{B} = \int \frac{\mu_B(y)}{y} \qquad x, y \in R, \quad \mu \in [0,1]$$

the $\tilde{A}*\tilde{B}$ ($*$ stands for either one of these operators +,−,x) defined as

$$\tilde{A}*\tilde{B} = \int \frac{\mu_A(x) \wedge \mu_B(y)}{x*y}$$

$$A^n = \int \frac{\mu_A(x)}{x^n}$$

$$A^{1/n} = \int \frac{\mu_A(x)}{x^{1/n}}$$

Definition 3. Let U,V be two universes and define

$$\tilde{M} : UxV \rightarrow F(R)$$

as R-fuzzy relation between U and V, or R-fuzzy matrix expressed as

$$\tilde{M} = (\tilde{m}_{ij})_{1xk}, \qquad \tilde{m}_{ij} \in F(R)$$

Definition 4. Let $\tilde{D}$, $\tilde{E}$ and $\tilde{G}$ be fuzzy metrices, the operator $*$ (+,−) is defined as follows

$$\tilde{D}*\tilde{E} = (\tilde{d}_{ij})_{mxn} * (\tilde{e}_{ij})_{mxn}$$

$$= (\tilde{d}_{ij} * \tilde{e}_{ij})_{mxn}$$

$$\tilde{E} \cdot \tilde{F} = (\tilde{e}_{ij})_{mxn} \cdot (\tilde{f}_{ij})_{mxl}$$

$$= (\sum_{k=1}^{n} \tilde{e}_{ik} \tilde{f}_{kj})_{mxl}$$

Definition 5. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \in F(R)$

$$x = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$$

is defined as R-fuzzy vector.

Definition 6.Let $\tilde{R}$ is R-fuzzy matrix, $\tilde{x}$ be R-fuzzy vector, if there exists a $\tilde{\lambda} \in F(R)$ such that

$$\tilde{R} \cdot \tilde{x} = \tilde{\lambda} \cdot \tilde{x}$$

holds, then $\tilde{\lambda}$ is called the eigenvalue of $\tilde{R}$ and the R-fuzzy vector is called the eigenvector for $\tilde{\lambda}$.

III. The fundamentals of fuzzy AHP

Let the goal be B, and the alternative set be A

$$A = (a_1, a_2, \ldots, a_n)$$

there is a R-fuzzy set

$$\underset{\sim}{A} = \sum_{i=1}^{n} \frac{\mu_A(a_i)}{a_i} , \qquad \mu_A(a_i) \in F(R)$$

Let the criterion set

$$P = \{P_1, P_2, \ldots, P_m\}$$

where $P_i = (P_{i1}, P_{i2}, \ldots, P_{ik})$, $i = 1,2,\ldots,m$, then, we have R-fuzzy matrices

$$\underset{\sim}{M}_{BP_1} : P_1 x P_1 \longrightarrow F(R)$$

$$\underset{\sim}{M}_{BP_1} = (\underset{\sim}{m}_{ij})_{k_1 x k_1} , \qquad \underset{\sim}{m}_{ij} \in F(R)$$

$$\underset{\sim}{M}_{P_i P_{i+1}} : P_{i+1} x P_{i+1} \longrightarrow F(R)$$

$$\underset{\sim}{M}_{P_i P_{i+1}} = (\underset{\sim}{m}_{ij})_{k_{i+1} x k_{i+1}} , \quad \underset{\sim}{m}_{ij} \in F(R), \quad i=1,2,\ldots,m$$

$$\underset{\sim}{M}_{P_m A} : AxA \longrightarrow F(R)$$

$$\underset{\sim}{M}_{P_m A} = (\underset{\sim}{m}_{ij})_{k_m x k_m} , \qquad \underset{\sim}{m}_{ij} \in F(R)$$

Their largest fuzzy eigenvalue is

$$\underset{\sim}{\lambda}_{BP_1}, \underset{\sim}{\lambda}_{P_i P_{i+1}} , \quad (i = 1,2,\ldots,m), \underset{\sim}{\lambda}_{P_m A}$$

respectively, and the corresponding eigenvectors are

$$\underset{\sim}{x}_{BP_1}, \underset{\sim}{x}_{P_i P_{i+1}} \quad (i =1,2,\ldots,m), \underset{\sim}{x}_{P_m A}$$

Then the result of the ordering the alternatives is

$$\underset{\sim}{x}_{P_m A}, \underset{\sim}{x}_{P_{m-1} P_m} \cdots \underset{\sim}{x}_{P_2 P_1} = (\mu_A(a_1), \ldots, \mu_A(a_n))$$

We can prove that the AHP is a special case of fuzzy AHP.

IV. Multiple experts

In fact, we need multiple experts to rank the alternatives in the AHP. Suppose that the n expert's evaluations are

$$E_1, E_2, \ldots, E_n$$

where

$$E_i = [\alpha_i, \beta_i] \quad \alpha_i \cdot \beta_i \in R, \qquad i= 1,2,\ldots,n$$

Then, the accumulation of these evaluations can form a distribution as shown in Fig. 1 which can be expressed as follows

$$x(u) = \frac{1}{n} \sum_{i=1}^{n} x_{[\alpha_i, \beta_i]}(u) , \qquad u \in R$$

where

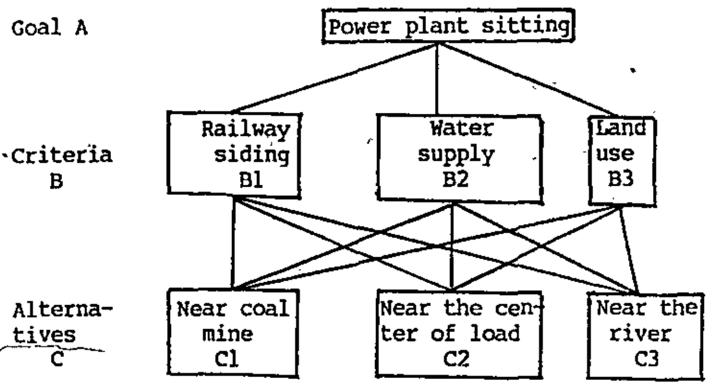
$$x_{[\alpha_i, \beta_i]}(u) = \begin{cases} 1 & \text{if } \alpha_i \leq u \leq \beta_i \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the n expert's evaluations can be expressed in a R-fuzzy set.

Fig. 1

V. Application

There is a power plant sitting problem, the structure of AHP is



Let

$$\underset{\sim}{1} = 0.5/0.9 + 1/1 + o.5/1.1$$

$$\underset{\sim}{2} = 0.5/1,9 + 1/2 + 0.5/2.1$$

$$\underset{\sim}{3} = 0.5/2.9 + 1/3 + 0.5/3.1$$

$$\underset{\sim}{4} = 0.5/3.9 + 1/4 + 0.5/4.1$$

$$\underset{\sim}{5} = 0.5/4.9 + 1/5 + 0.5/5.1$$

$$\underset{\sim}{6} = 0.5/5.9 + 1/6 + 0.5/6.1$$

$$\underset{\sim}{7} = 0.5/6.9 + 1/7 + 0.5/7.1$$

We have the R-fuzzy matrices

A-B

A	B1	B2	B3
B1	$\underset{\sim}{1}$	$\underset{\sim}{2}$	$\underset{\sim}{4}$
B2	$\underset{\sim}{1/2}$	$\underset{\sim}{1}$	$\underset{\sim}{4}$
B3	$\underset{\sim}{1/4}$	$\underset{\sim}{1/4}$	$\underset{\sim}{1}$

B1-C

B1-C	C1	C2	C3
C1	$\underset{\sim}{1}$	$\underset{\sim}{7}$	$\underset{\sim}{5}$
C2	$\underset{\sim}{1/7}$	$\underset{\sim}{1}$	$\underset{\sim}{1/2}$
C3	$\underset{\sim}{1/5}$	$\underset{\sim}{2}$	$\underset{\sim}{1}$

B2-C

B2	C1	C2	C3
C1	$\underset{\sim}{1}$	$\underset{\sim}{1/4}$	$\underset{\sim}{1/3}$
C2	$\underset{\sim}{4}$	$\underset{\sim}{1}$	$\underset{\sim}{2}$
C3	$\underset{\sim}{3}$	$\underset{\sim}{1/2}$	$\underset{\sim}{1}$

B3-C	C1	C2	C3
C1	$\underset{\sim}{1}$	$\underset{\sim}{1}$	$\underset{\sim}{5}$
C2	$\underset{\sim}{1}$	$\underset{\sim}{1}$	$\underset{\sim}{5}$
C3	$\underset{\sim}{1/5}$	$\underset{\sim}{1/5}$	$\underset{\sim}{1}$

After obtaining the fuzzy eigenvalues for each R-fuzzy matrix, we have the fuzzy eigenvector as

$$\underset{\sim}{W}_A = (\underset{\sim}{W}_{A1}, \underset{\sim}{W}_{A2}, \underset{\sim}{W}_{A3})^T$$

where

$$\underset{\sim}{W}_{A1} = \frac{0.5}{0.5448} + \frac{1}{0.5494} + \frac{0.5}{0.5497}$$

$$\vdots$$

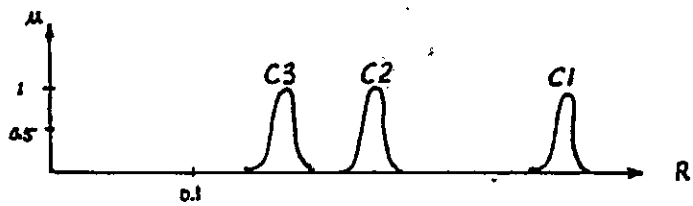
The result is

$$\underset{\sim}{W}_{c1} = \frac{0.5}{0.4965} + \frac{1}{0.4969} + \frac{0.5}{0.5034} \cdot$$

$$\underset{\sim}{W}_{c2} = \frac{0.5}{0.2883} + \frac{1}{0.2916} + \frac{0.5}{0.2949}$$

$$\underset{\sim}{W}_{c3} = \frac{0.5}{0.2074} + \frac{1}{0.2108} + \frac{0.5}{0.2136}$$

We can see that the best alternative is C3. See Fig 2.



VI. Conclusion

In this paper, we have presented a concrete application of R-fuzzy set theory to an analytical hierarchical process. In the previous research, Van Learhoven and Pedrycz (3) extend Saaty's hierarchical analysis to fuzzy AHP by using fuzzy numbers which are only the triangular fuzzy numbers. J. J. Buckley's method (4) is to substitute the fuzzy ratio into the solution of normal equations. He summarized the judgements of the experts by using geometric means. In our method, R-fuzzy set was used instead of fuzzy number. In fact, fuzzy number is a kind of R-fuzzy set while we define some simple operators on R-fuzzy set, so that the calculation can be done simply. The more important fact is that the expert's evaluations can be formed in a R-fuzzy set, it also can reduce the calculation process. Therefore, it can be easily used in practice. Lastly, we can see that our method is an extension of AHP.

REFERENCES

1. Thomas L. Saaty, The Analytic Hierarchy Process, McGraw-Hill, New York 1980.
2. J. G. de Graan, Extension to the Multiple Criteria Analysis Method of T. L. Saaty, National Institute For Water Supply, Voorburg,Netherlands, 1980.
3. P. J. M. van Laarhoven and W. Pedrycz, A Fuzzy Extension of Saaty's Priority Theory, Fuzzy sets and Systems, 1983.
4. J. J. Buckley, Fuzzy Hierarchical Analysis, Fuzzy sets andSystem, 1985.
5. J. J. Buckley, Ranking Alternatives Using Fuzzy Numbers, Fuzzy sets and Systems, 1985.
6. Hu Zhao-guang, R-fuzzy Sets of Type 2 and its Application to Decision Making, ISEMA 88.
7. D. Dubois and H. Prade, Fuzzy sets and Systems, Academic Press, New York, 1980.