THE CONSTRUCTION OF FUZZY JUDGMENT MATRIX AND ITS RANK—ORDERING IN AHP

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ABSTRACT

In this paper, starting from the angle of actual application, considering the method of group decision—making in complex systems and the fuzziness of judgment in pairwise comparison of projects, we construct the fuzzy judgment matrix by using set—valued statistics method on continuous judgment scale and prove that every element of the fuzzy judgment matrix can be represented by positive bounded closed fuzzy number. After operated properties of positive bounded closed fuzzy number are discussed, the fuzzy weight vector of projects can be introduced by applied fuzzy extension principle to the altered gradient eigenvector method. Thus, the rank—ordering of projects is given. In this paper, we also give an example to apply our model.

INTRODUCTION

In AHP, the relative importance of a set of projects are determined by pairwise comparison. That is efficient for dealing with the rank—ordering problems with relative properties and without certain identical scale can be used. But when we assign a number to represent the result of pairwise comparison, we deny all the other numbers. That is not so good because people's thinking and judgment remain fuzziness and sometimes we also can not make out a certain number for the pairwise comparison especially in complex systems. Van Larrhoven and W. Pedrycz presented a method in which fuzzy number with triangular membership functions were used to represent fuzzy judgment [3]. They assumed that the result of pairwise comparison of projects can be represented as fuzzy numbers with triangular membership functions. That is convenient for operation, but the problems of the construction of fuzzy number with triangular membership function and its objectivism should be researched further. In this paper, starting from the angle of actual application, considering the method of group decision—making in complex systems and the fuzzinss of judgment in pairwise comparison of projects, we construct the fuzzy judgment matrix by using set—valued statistics method on continuous judgment scale and prove that every element of the fuzzy judgment matrix can be represented by positive bounded closed fuzzy number. After discussing the operated properties of positive bounded closed fuzzy number, we can extract the fuzzy weight vector of projects from fuzzy judgment matrix. Thus, the rank—ordering of projects is given by corresponding fuzzy weights.

PRELIMINARIES

In order to discuss the problem of the construction of fuzzy judgment matrix and its rank-ordering, we have to solve the operation properties of positive bounded closed fuzzy number first. Let R be real number fields, $R^+ = \{ x; x>0, x \in R \}$.

Definition 1. Let $F(R) = \{ \underset{\sim}{A}: \underset{\sim}{A}$ is fuzzy subset on $R \}$. $\underset{\sim}{A} \in F(R)$, if
(1) $\exists x' \in R$, such that $\underset{\sim}{A}(x') = 1$
(2) $\forall \lambda \in (0,1]$, $A_\lambda = \{ x; \underset{\sim}{A}(x) \geq \lambda \}$ is a convex set
Then the fuzzy subset $\underset{\sim}{A}$ is called fuzzy number on R.

Definition 2. Let $\underset{\sim}{A}$ be fuzzy number on R^+, if $\forall \lambda \in (0,1]$, A_λ is a closed set, then $\underset{\sim}{A}$ is called positive closed fuzzy number; if A_λ is also a bounded set, then A is called positive bounded closed fuzzy number. Let $F(R^+) = \{ \underset{\sim}{A}: \underset{\sim}{A}$ is positive bounded closed fuzzy number on $R^+ \}$

From Definition 2, we can get the result that if $\underset{\sim}{A} \in F(R^+)$, then $\forall \lambda \in (0,1]$, A_λ can be represented as a closed interval. This is because A_λ is a bounded closed convex set, let $a = \inf A_\lambda$, $b = \sup A_\lambda$, we can prove $A_\lambda = [a,b]$.

According to fuzzy extension principle [1], we can get the operation principle of positive closed interval.

Theorem 1. Let $[a,b]$, $[c,d]$ be positive closed interval, $c>0$, then
(1) $[a,b] + [c,d] = [a+c, b+d]$
(2) $[a,b] \cdot [c,d] = [ac, bd]$
(3) $[a,b] / [c,d] = [a/d, b/c]$
(4) $c[a,b] = [ca, cb]$
(5) $1/[a,b] = [1/b, 1/a]$

For the operations of positive bounded closed fuzzy number, by fuzzy extension principle [1], we have following theorem;

Theorem 2. Let $\underset{\sim}{A}, \underset{\sim}{B} \in F(R^+)$, $A_\lambda = [s_\lambda, r_\lambda]$, $B_\lambda = [p_\lambda, q_\lambda]$, $c>0$, then
(1) $\underset{\sim}{A} + \underset{\sim}{B} = \bigcup_{0<\lambda\leq 1} \lambda (A_\lambda + B_\lambda) = \bigcup_{0<\lambda\leq 1} \lambda [s_\lambda + p_\lambda, r_\lambda + q_\lambda]$
(2) $\underset{\sim}{A} \cdot \underset{\sim}{B} = \bigcup_{0<\lambda\leq 1} \lambda (A_\lambda \cdot B_\lambda) = \bigcup_{0<\lambda\leq 1} \lambda [s_\lambda p_\lambda, r_\lambda q_\lambda]$
(3) $\underset{\sim}{A} / \underset{\sim}{B} = \bigcup_{0<\lambda\leq 1} \lambda (A_\lambda / B_\lambda) = \bigcup_{0<\lambda\leq 1} \lambda [s_\lambda / q_\lambda, r_\lambda / p_\lambda]$
(4) $c\underset{\sim}{A} = \bigcup_{0<\lambda\leq 1} \lambda (c\underset{\sim}{A})_\lambda = \bigcup_{0<\lambda\leq 1} [cs_\lambda, cr_\lambda]$ (where $c\underset{\sim}{A}$ is derived from function $f(x) = cx$ by extension principle)
(5) $1/\underset{\sim}{A} = \bigcup_{0<\lambda\leq 1} \lambda (1/A_\lambda) = \bigcup_{0<\lambda\leq 1} \lambda [1/r_\lambda, 1/s_\lambda]$

THE CONSTRUCTION OF FUZZY JUDGMENT MATRIX UNDER SINGLE CRITERION

Suppose that the set of elements in some level is $U=\{u1, u2, \ldots, u_n\}$. Under some criterion, our aim is to determine the rank-ordering of relative importance of every elements in the level. We use following method to construct the fuzzy judgment matrix under single criterion.

First, extending the 1-9 scale that presented by Saaty [2] to continuous scale interval (0, 10], that is, at 1, 3, 5, 7, 9, we have their original meaning, and we use other point x, $x \in (0, 10]$, to present the intermediate state, the bigger the value x is, the more obvious the importance comparison is. Suppose the decision-making group have persons m, for any $u_i, u_j \in U$, every decision-maker have to make out his judgment of importance comparison independently, and according to the scale described above, give out a range on interval (0, 10] to represent the importance comparison. Denote the range given out by the kth decision-maker is $\xi_k=[a_k, b_k]$, $(k=1, \ldots, m)$. Of course, the result maybe appear to have ξ_{k_0}, ξ_{l_0} such that $\xi_{k_0} \cap \xi_{l_0}=\Phi$, $(k_0, l_0 \in \{1, \cdots, m\})$, for this case, the result can be adjusted by exchanging their opinions between the kth decision-maker and the lth decision-maker, and finally the result: $\xi_k \cap \xi_l \neq \phi$ $(\forall k, l \in \{1, \ldots, m\})$ can be reached. Under this condition, we have following conclusion:

Theorem 3. Let $\xi_k \cap \xi_l \neq \phi (\forall k, l \in \{1, \cdots, m\})$, $\xi_k=[a_k, b_k]$ $(k=1, \cdots, m)$, then,

$$\underset{\sim}{a}_{ij}(x)=1/m\sum_{k=1}^{m}X_{\xi_k}(x) \qquad x \in (0, 1] \tag{1}$$

(X_{ξ_k} is ξ_k's characteristic function)
is a closed fuzzy number on (0, 10].

Proof. We can let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m$. Since $\forall k, l \in \{1, \cdots, m\}$, have $\xi_k \cap \xi_l = \phi$, then we have $a_k \leqslant b_l$ $(\forall k, l \in \{1, \cdots, m\})$. So 2m numbers $a_1, \cdots, a_m, b_1, \cdots, b_m$ can be ranked from small to big as following form:

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m \leqslant b_{i_1} \leqslant b_{i_2} \leqslant \cdots \leqslant b_{i_m}$$

In the case of $x \in (0, b_{i_1}]$, if $x \in [a_l, a_{l+1})$ $(l=1, 2, \cdots, m-1)$, we have $\underset{\sim}{a}_{ij}(x)=l/m$; if $x<a$, then $\underset{\sim}{a}_{ij}(x)=0$; and if $x \in [a_m, b_{i_1}]$, then $a_{ij}(x)=1$. Hence, in interval $(0, b_{i_1}]$, $\underset{\sim}{a}_{ij}(x)$ is a monotone non-decreasing right continuous ladder function. Similarly, in interval $[a_m, 10]$, $\underset{\sim}{a}_{ij}(x)$ is a monotone non-increasing left continuous ladder function. So $\forall \lambda \in (0, 1]$, $(\underset{\sim}{a}_{ij})_\lambda=\{x: \underset{\sim}{a}_{ij}(x) \geqslant \lambda\}$ is a closed interval, namely, $(\underset{\sim}{a}_{ij})_\lambda$ is a convex set, because of $\underset{\sim}{a}_{ij}(x)=1$ for $x \in [a_m, b_{i_1}]$, by Definition 1, $\underset{\sim}{a}_{ij}(x)$ is a closed fuzzy number on (0, 10].

In the level $U=\{u_1, \cdots, u_n\}$, there are $n(n-1)/2$ judgments required to be made. When we compare the importance of u_i u_j $(i<j)$, if u_j's importance is stronger than that of u_i, according to the method mentioned above, we can first get $\underset{\sim}{a}_{ji}(x)$, by (5) in Theorem 2, we have $\underset{\sim}{a}_{ij}(x)=1/\underset{\sim}{a}_{ji}(x)$. So the $n(n-1)/2$ judgmments can be represented by a upper triangular matrix as

follows:

$$A=\begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ & 1 & a_{23} & \cdots & a_{2n} \\ & & & \cdots & \\ & & & & 1 \end{bmatrix}_{n\times n}$$

where diagonal elements represent the importance comparison of u_i with itself, so $a_{ii}=1$ ($i=1,\cdots,n$), and each element a_{ij} ($i<j$) is bounded closed fuzzy number and is Lebesgue measurable.

LOCAL FUZZY PRIORITIES FROM UPPER TRIANGULAR FUZZY JUDGMENT MATRIX

For the rank-ordering problem of upper triangular judgment matrix

$$A=\begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ & 1 & a_{23} & \cdots & a_{2n} \\ & & & \cdots & \\ & & & & 1 \end{bmatrix}_{n\times n}$$

the altered gradient eigenvector method can deal with this problem [5] i.e.

$$D=\begin{bmatrix} n & & \\ & n-1 & \\ & & 1 \end{bmatrix}_{n\times n} \qquad B=[\,b_{ij}\,]_{n\times n}$$

where

$$b_{ij}=\begin{cases}(n-1)\lambda_{ij} & i<j\\ 1 & i=j\\ 0 & i>j\end{cases} \qquad \lambda_{ij}\geqslant 0 \qquad \sum_{j=i+1}^{n}\lambda_{ij}=1$$

λ_{ij} are determined by actual problem.

From the eigenvalue problem

$$D^{-1}(A\circ B)W=W \qquad (\text{ where } \text{“}\circ\text{”} \text{ is Hadamard product })$$

we can derive

$$w_i=\sum_{j=i+1}^{n}\lambda_{ij}a_{ij}w_j \qquad (i=1,\ldots,n-1) \qquad (2)$$
$$w_n>0 \text{ can be assigned arbitrarily}$$

where $(w_1,\cdots,w_n)$ is weight vector that corresponds to $(u_1,\cdots,u_n)$. This method is easy for operation, and the parameters λ_{ij} can be adjusted by people's experience, in the case of consistency (namely $a_{ij}=a_{ik}a_{kj}$), the result is the same as that from Eigenvalue Method. So this method could be used as a basis for our investigation.

For the upper triangular fuzzy judgment matrix

$$A = \begin{bmatrix} 1 & \underset{\sim}{a}_{12} & \underset{\sim}{a}_{13} & \cdots & \underset{\sim}{a}_{1n} \\ & 1 & \underset{\sim}{a}_{23} & \cdots & \underset{\sim}{a}_{2n} \\ & & \cdots\cdots & & \\ & & & & 1 \end{bmatrix}_{n\times n}$$

applying fuzzy extension principle [1] to equation (2), we have

$$\underset{\sim}{w}_i=\sum_{j=i+1}^{n}\lambda_{ij}\underset{\sim}{a}_{ij}\underset{\sim}{w}_j \qquad (i=1,\cdots,n-1) \qquad (3)$$
$\underset{\sim}{w}_n > 0$ can be assigned arbitrarily.

where $\underset{\sim}{w}_i$ ($i=1,\cdots,n$) are the fuzzy weights that correspond to the relative importance of u_i. Go into details, we have

$$\underset{\sim}{w}_i=\bigcup_{0<\lambda\leq 1}\lambda(\underset{\sim}{w}_i)_\lambda=\bigcup_{0<\lambda\leq 1}\lambda\left[\sum_{j=i+1}^{n}\lambda_{ij}(\underset{\sim}{a}_{ij})_\lambda(\underset{\sim}{w}_j)_\lambda\right] \quad (i=1,\cdots,n-1) \qquad (4)$$

Let $\underset{\sim}{w}_n$ be positive bounded closed fuzzy number, since $\underset{\sim}{a}_{ij}$ are all positive bounded closed fuzzy numbers, by Theorem 2, we have $w_{n-1},w_{n-2},\cdots,w_1$ are all positive bounded closed fuzzy numbers. Let

$$(\underset{\sim}{a}_{ij})_\lambda=[(\underset{\sim}{a}_{ij})_\lambda^1,(\underset{\sim}{a}_{ij})_\lambda^2] \qquad (\underset{\sim}{w}_j)_\lambda=[(\underset{\sim}{w}_j)_\lambda^1,(\underset{\sim}{w}_j)_\lambda^2]$$

we have

$$(w_i)_\lambda=\sum_{j=i+1}^{n}\lambda_{ij}(\underset{\sim}{a}_{ij})_\lambda(\underset{\sim}{w}_j)_\lambda$$
$$=\left[\sum_{j=i+1}^{n}\lambda_{ij}(\underset{\sim}{a}_{ij})_\lambda^1(\underset{\sim}{w}_j)_\lambda^1,\sum_{j=i+1}^{n}\lambda_{ij}(\underset{\sim}{a}_{ij})_\lambda^2(\underset{\sim}{w}_j)_\lambda^2\right] \qquad (5)$$

where $\lambda_{ij}\geq 0$, $\sum_{j=i+1}^{n}\lambda_{ij}=1$, λ_{ij} is weight that corresponds to $\underset{\sim}{a}_{ij}$. By equations (4), (5), if λ_{ij} are determined; we can calculate the fuzzy priorities, and hence, get the rank-ordering of elemments $u_1,\cdots,u_n$. Because λ_{ij} are the weights that correspond to a_{ij}, when we assign a number to λ_{ij}, we have to consider the properties of $\underset{\sim}{a}_{ij}$ itself. On $\lambda\in(0,1]$ level, the bigger the Lebesgue measure of $(\underset{\sim}{a}_{ij})_\lambda$ is, the smaller confidence extent of importance comparison of u_i, u_j it means, correspondingly, we must assign a smaller number to λ_{ij}. In this way, we can let different confidence extent play different part in the decision-making process. Generally speaking, if $L((\underset{\sim}{a}_{ij})_\lambda)=0$ (L represents Lebesgue measure), it means that the fuzzy number which have been assigned to $\underset{\sim}{a}_{ij}$ could be completely confidence on the λ level. If there are s ($0<i<n+1-s$) members in $\{(a_{i\,i+1}),\cdots,(a_{in})\}$ which Lebesgue measure are zero, then the corresponding are assigned as $1/s$; if $L((\underset{\sim}{a}_{ij})_\lambda)\neq 0$, then the corresponding λ_{ij} are assigned as

$$\lambda_{ij}=[1/L((\underset{\sim}{a}_{ij})_\lambda)]/\left[\sum_{j=i+1}^{n}[1/L((\underset{\sim}{a}_{ij})_\lambda)]\right] \qquad (1<i<n-2,\ i<j) \qquad (6)$$

So far, we can calculate $\underset{\sim}{w}_i$ ($i=1,\ldots,n$), especially, we can normalize w_i ($i=1,\ldots,n$). Let

$$\tilde{w}_i = w_i / \sum_{j=1}^{n} w_j \qquad (7)$$

we have

$$\tilde{w}_i = \bigcup_{0\le\lambda\le1} \lambda \left[(w_i)^1_\lambda / \sum_{j=1}^{n} (w_j)^2_\lambda ,\ (w_i)^2_\lambda / \sum_{j=1}^{n} (w_j)^1_\lambda \right] \qquad (8)$$

SYNTHESIS OF PRIORITIES

Suppose $w_{1i}, w_{2i}, \cdots, w_{ni}$ are the local fuzzy priorities that correspond to elements $u_1, \cdots, u_n$ under the criterion A^i $(i=1, \cdots, q)$, $a_1, \cdots, a_q$ are the overall fuzzy priorities that correspond to $A^1, \cdots, A^q$. Then the overall fuzzy priorities of elements $u_1, \cdots, u_n$ can be represented as follows

$$w_i = \sum_{j=1}^{q} a_j w_{ij} \quad , \quad (i=1, \cdots, n) \qquad (9)$$

Let

$$(a_j)_\lambda = [(a_j)^1_\lambda, (a_j)^2_\lambda] \qquad (w_{ij})_\lambda = [(w_{ij})^1_\lambda, (w_{ij})^2_\lambda]$$

then

$$\begin{aligned}(w_i)_\lambda &= \sum_{j=1}^{q} (a_j)_\lambda (w_{ij})_\lambda \\ &= \left[\sum_{j=1}^{q} (a_j)^1_\lambda (w_{ij})^1_\lambda ,\ \sum_{j=1}^{q} (a_j)^2_\lambda (w_{ij})^2_\lambda\right] \end{aligned} \qquad (10)$$

and hence

$$w = \bigcup_{0\le\lambda\le1} \lambda \left[\sum_{j=1}^{q} (a_j)^1_\lambda (w_{ij})^1_\lambda ,\ \sum_{j=1}^{q} (a_j)^2_\lambda (w_{ij})^2_\lambda\right] \qquad (11)$$

This method for synthesis of priorities reflects that every local fuzzy priorities play a same part in the calculation of overall fuzzy priorities. Using the same method, for every level in the hierarchy, overall fuzzy priorities can be composited from the second level to the bottom level which contains the set of projects to be chosen. Finally, the global fuzzy priorities that correspond to the bottom level can be gotton. According to the global fuzzy priorities, we can make the choice.

AN EXAMPLE

Suppose that there are three colleges, called B1, B2, B3 respectively. In order to evaluate their standard of running a school, a committee, including three members, has been installed to give the rank-ordering of them. The decision criteria have been made out as follows:

A1: the effect in teaching and researching
A2: the conditions of running a school and input-output benifit
A3: administrative level

Apply the method presented above. The results are as follows:

The upper triangular fuzzy judgment matrix of the criteria together with its fuzzy weight vector are

	A1	A2	A3	weight vector
A1	1	[4·5, 5·5] [4·0, 5·8] [3·5 6·0]	[1·8, 2·1] [1·6, 2·2] [1·5 2·4]	[0·50022, 0·68237] [0·41888, 0·78607] [0·35851, 0·90828]
A2		1	[0·32, 0·36] [0·3, 0·38] [0·28, 0·4]	[0·09323, 0·11854] [0·08378, 0·13571] [0·07368, 0·15138]
A3			1	[0·29035, 0·32928] [0·27925, 0·35714] [0·26316, 0·37845]

Denote a1 a2 a3 b1 b2 b3 as [a3, b1] [a2, b2] [a1, b3]

The upper triangular fuzzy judgment matrix of colleges for each criterion together with their local fuzzy weight vectors are as follows:

A1	B1	B2	B3	weight vector
B1	1	[0·95, 1·08] [0·9, 1·15] [0·82, 1·2]	[2·0, 2·1] [1·8, 2·2] [1·75, 2·24]	[0·37635, 0·44484] [0·30557, 0·53000] [0·19094, 0·84939]
B2		1	[1·8, 1·9] [1·7, 2·05] [1·65, 2·1]	[0·36515, 0·40651] [0·31794, 0·47302] [0·25835, 0·54271]
B3			1	[0·20084, 0·21395] [0·18702, 0·23074] [0·15657, 0·58430]

A2	B1	B2	B3	weight vector
B1	1	[0·73, 0·8] [0·7, 0·84] [0·6, 0·9]	[1·5, 1·8] [1·4, 1·9] [1·2, 2·2]	[0·30734, 0·38203] [0·26376, 0·44496] [0·20304, 0·54493]
B2		1	[2·1, 2·2] [2·0, 2·35] [1·85, 2·4]	[0·42274, 0·47550] [0·37680, 0·53409] [0·33218, 0·60290]
B3			1	[0·20131, 0·21613] [0·18840, 0·22727] [0·17956, 0·25121]

A3	B1	B2	B3	weight vector
B1	1	[0·73, 0·8] [0·7, 0·84] [0·6, 0·9].	[1·5, 1·8] [1·4, 1·9] [1·2, 2·2]	[0·30734, 0·38203] [0·26376, 0·44496] [0·20304, 0·54493]
B2		1	[2·1, 2·2] [2·0, 2·35] [1·85, 2·4]	[0·42274, 0·47550] [0·37680, 0·53409] [0·33218, 0·60290]
B3			1	[0·20131, 0·21613] [0·18840, 0·22727] [0·17956, 0·25121]

The global fuzzy weight vector is

B1	B2	B3
[3.8846 , 0.56478]	[0.26746, 0.39322]	[0.17825, 0.24157]
[0.28574, 0.75902]	[0.20191, 0.51985]	[0.14721, 0.29616]
[0.19752, 1.18781]	[0.14811, 0.67317]	[0.17708, 0.36640]

According to the global fuzzy weight vector, it will be clear that the priority queue is B1, B2, B3.

CONCLUSION

There are three characters in this paper. The first, let decision－maker give out the range of pairwise comperison judgment of elements on continuous judgment scale, that is easy for decision－maker, and can be accepted and be applied. The second, by using the version of nested sets, the operations can be carried on in every level and through the operations of endpoints of closed interval. So the complex question can be changed to simple one for solving. The third, in choice of weight λ_{ij} of g_{ij} , consider that different confidence extent of judgment must play different part in calculation of fuzzy weight, so we can easily introduce the experience and thinking judgment into the rank－ordering of projects.

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