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TWO DECISION PATTERNS OF THE AHP AND RANK REVERSAL

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Abstract: *When one employs the AHP to make decisions, it is important to identify two kinds of decision problems. One is the multiple attribution decision making, the other is the multiple criteria decision making. Two patterns of the AHP: WEIGHT DISTRIBUTION PATTERN AND WEIGHT REFERENCE PATTERN should be applied. In this paper, the definitions and characteristics of the two decision patterns for deriving the global priorities of alternatives are presented. Specifically, the REFERENCE SYSTEM for priority is emphasized in a detailed discussion. According to the two decision patterns, the controversy of the rank reversal is reviewed.*

Introduction

It has been fourteen years since T. L. Saaty introduced the Analytic Hierarchy Process[4], a useful method for measurement of intangible factors and decision making. Many of the AHP applications, including recourse allocation, alternative evaluation, economic planning, conflict resolution and so on, have been explored around the world[7]. People has a good reason to believe that the AHP is an effective method to deal with decision making problems in a complex environment. On the other hand, some criticisms, especially concerning the rank reversal phenomenon of the AHP, have been raised since 1982[1]. It is always true in the history of science that a new idea needs to be improved and consummated after its introduction. Through critics and anti-critics a scientific theory will be developed to be more perfect. We believe that the AHP is such a case.

The phenomenon of the rank reversal can be defined by the following statement:

> RANK REVERSAL: When a set of alternatives in the lower level have been ranked separately on each of several elements from the high level and their overall ranks with respect to the decision goal are derived, a new alternative which is added to the collection will change the old overall ranking.

Dyer[2] properly pointed out that the nature of the rank reversal phenomenon rests with the normalization process. The questions raised were: is the rank of alternatives by AHP arbitrary, is the rank reversal phenomenon absolutely flawless and will the phenomenon turn out to be an insuperable barrier if the utility theory is not used to rebuild the AHP?

In fact, for some decision problems the rank reversal is a reasonable phenomenon using the AHP to derive the overall priorities for alternatives. For other decision problems the rank reversal can be avoided by adjusting the procedure of the AHP. When one makes decision using the AHP, it is necessary to differentiate between multiple attribute and criteria decision making problems. Attributes are often refered to as differentiating aspects, properties or characteristics of an alternative while criteria are generally considered as valuable measures, dimensions or scales against which alternatives may be gauged in a value or worth sense. In this paper, we present the basic idea of dividing the methodology of the AHP into two patterns: distribution pattern and reference pattern according to multiple attribute and multiple criteria decision making problems, respectively. Section II and III briefly explain definitions, characteristics, deriving priorities of alternatives and application ranges of the multiple attribution decision problem and multiple criteria decision problem, respectively. The procedures of deriving the overall priorities by using the AHP for the multiple criteria decision problem, i.e. the reference pattern of the AHP, are presented in details in Section IV. Then the rank reversal phenomenon based on the concepts of the two patterns is discussed. The condusion for the questions mentioned above is provided in Section VI.

To avoid a long and tedious exposition, assume that a simple hierarchical structure for a decision problem is employed which includes three levels: the decision goal level, the attribute or criterion level and an alternative level. The result introduced by the simple structure can be switched to a more complex structure without any difficulty. The following symbols will be used through this paper:

- C_i : the ith attribute or criterion, i = 1, 2, ..., m
- c_i : the weight of the attribute or criterion i with respect to the decision goal,
- A_j : the jth alternative, j = 1, 2, ..., n,
- w_{ij}: the weight of the alternative j with respect to the attribute or criterion i,
- w_j : the weight of the alternative j with respect to the decision goal.

In addition, we assume that readers are familiar with the basic principle and procedure of the AHP[5].

Multiple Attribute Decision Making and the Distribution Pattern of the AHP

Suppose we are faced with the following decision problem: promoting a college faculty from a 4-member group from associate rank to full professor rank according to his contribution to teaching and research while working in the group. The problem can be structured by the hierarchy in Figure 1

In this tree structure the evaluation goal is broken into two attributes: teaching and research. Then the teaching and research are broken into the contributions made by each of the four faculties, i.e. A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}, A_{41}, A_{42}. Even though the tree structure can be simplified by the hierarchy structure as shown in Figure 2, there are different meanings of the alternatives when we make the pairwise comparisons of the alternatives with respect to each attribute.

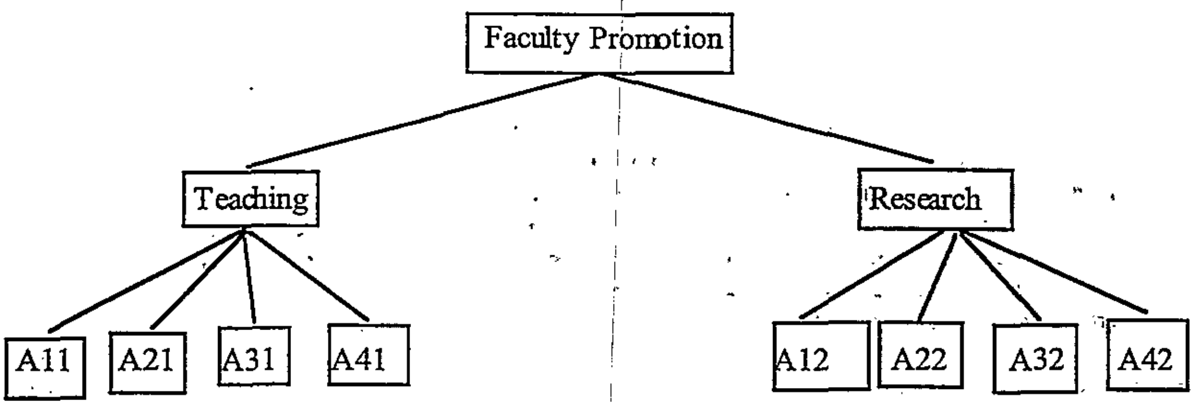

Figure 1

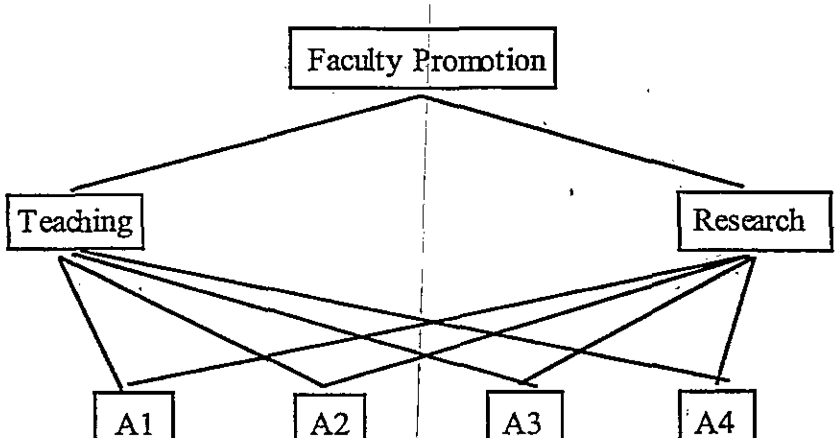

Figure 2

After the weights of the two attributes have been determined, we can find the weights of each faculty with respect to the attributes by distributing the weight of each attribute according to those relative contributions by each faculty to the attributes. Then the overall priorities of the faculty can be derived by the hierarchy composition principle. Obviously, the hierarchy composition principle for this example appears to be the process in which the weights of the criteria are distributed into the alternatives by their relative contribution to the attributes, i.e. the weights of alternatives are obtained by sharing the weights of attributes.

This is a simple example of a multiple attribute decision problem. The tree hierarchy structure can be used to express the decision problem. The element in the higher level is decomposed into the elements in lower level. The "belong to" relationship exists between the elements of the higher and lower level. The weights of the elements in higher level are distributed to the elements along the tree structure. The hierarchy decomposition principle of the AHP is suitable for the multiple attribute decision making problems. The pattern of the AHP is referred to as the distribution pattern.

To understand the distribution pattern of the AHP, the characteristics of the multiple attribute decision making problem must be analyzed. First of all, there exists an explicit inclusion relationship between elements of higher and lower levels. Thus, a tree structure is an inherent expression for structuring these types of decision problems. The weight of an element in the higher level can be determined without referencing the certain element in the lower level, i. e. the alternative does not affect determining the weight of the attribute which the alternative belongs to. The weight of an attribute is distributed over the alternatives which belong to the attribute according to their relative importance with respect to the attribute. The regular ratio scale and pairwise comparison should be used to obtain the normalized priorities of the alternatives. The hierarchy decomposition principle properly expresses the process of the weight distribution of attribution over alternatives. The rank reversal phenomenon may occur when a new alternative is added because the new alternative changes the distribution of the attribute weights over the alternatives. It should be a reasonable phenomenon and may be useful for multiple attribute decision making.

The regular procedure of the AHP should be used in multiple attribute decision making. It includes the following points:

1. Structuring the decision problem by a tree or hierarchy;
2. Making pairwise comparisons of the elements in a lower level with respect to their relative element in a higher level using 1-9 ratio scale. Entering the comparisons into a positive reciprocal matrix.
3. Calculating the normalized eigenvector of the reciprocal matrix which stands for the priority of the elements in the lower level with respect to the element in the higher level.
4. Deriving the overall priority of alternatives by the hierarchy decomposition principle.

We referred to the regular procedure of the AHP as the distribution pattern to distinguish from the reference pattern of the AHP.

Multiple Criteria Decision Making

Let us consider another faculty example similar to the example in section II. Suppose that after campus interviews we need to select one of the four applicants to fill the faculty position according to their teaching and research reputations. Even though the hierarchy structure in Figure 2 can be used to express the decision problem, the meanings of the elements of the structure are different. Here C_1 and C_2 are evaluation criteria of teaching and research for the applicants, respectively. The four alternatives stand for the comprehensive level of each applicant in teaching and research.

This is a typical example of a multiple criteria decision making problem. Note that there are several differences between multiple attribute and multiple criteria decision making when the AHP is applied for these two types of problems.

The first difference is the decision structure. In the structuring of multiple attibute decision problems, the elements in a lower level do not belong to the element in a higher level, i.e. there is no "including" relationship between the elements in lower and higher level. Therefore, the tree structure is no longer an expression for these decision problems in general.

Secondly, to determine the weights of the criteria by pairwise comparisons, one must refer to an alternative which is relative to the criteria as a standard. The reason for an alternative being used as a standard is that the pairwise comparison without a reference standard does not have an explicit meaning. For example, how can one answer the question: "which one criterion is more important with respect to the goal of evaluation of applicants, teaching or research? " without keeping a reference standard in his mind We always use an alternative or other standards as a reference point for answering the question even though the reference point is not explicated. The choice of the reference, an alternative, for example, may affect the derived weights of the criteria, because the reference point is applied when we make pairwise comparison to derive the weights of the criteria.

Thirdly, the weight of an alternative with respect to a criterion is determined by its relative importance to the reference point of the criterion if pairwise comparisons are applied. Therefore, the overall priority of an alternative is obtained by transition of the weights of the criteria which the alternative are relative to. The overall priority of an alternative can not be obtained by the distribution process of the weights of the criteria. In this case, we can not use the hierarchy decomposition principle and the normalization of the weights of the alternatives to derive the overall priority of an alternative.

Fourth, the phenomenon should not occur for the decision problems if the proper pattern of the AHP is used to derive the priority. Note the determination of the derived weight of an alternative is not relative to other alternatives. If the reference point has been chosen, the weights of an alternative can be obtained by making comparison between the alternative and the reference point with respect to the criterion even though the pairwise comparisons among alternative may give more reasonable priority. In this case, adding a new alternative should not affect the rank of a set of old alternatives.

The AHP, especially its 1-9 ratio scale and the procedure to derive priority by pairwise comparison, is still an effective method for the multiple criteria decision problem. In the next section we will concentrate on the discussion of the weight reference pattern of the AHP which is a proper method to derive the priority for the multiple criteria decision making problem.

The Weight Reference Pattern of the AHP

According to the features of the multiple criteria decision problems which are different from the multiple attribute decision problems, the regular AHP procedure should be adjusted by the following points:

1. Choose an appropriate reference point to derive the weights of the criteria. The reference point can be an appropriate alternative, one scale from a set of standard scales or a point of a physical measurement system.

2. Instead of the normalized weights of alternatives with respect to a criterion, we set up the weight of the reference point as one.

3. It is not necessary to normalize the overall priority of alternatives.

The weight reference pattern of the AHP has two different methods to derive the overall priorities of the alternatives according to the chosen reference point. They are

1. The reference alternative method,
2. The reference point method.

The details of the procedures of the two methods above are provided below.

The procedure of the reference alternative method

The reference alternative method chooses an alternative as a reference point to determine the weights of the criteria. This method is suitable for the decision problems which meet the following condtions:

–there exists at least one alternative which is related to all criteria, so that the alternative can be chosen as a reference point for all criteria;

–the number of criteria is less than 9, i.e. there is only an acceptable number of pairwise comparisons to derive the weights of the criteria.

If the above condtions are not met, other methods must be considered.

The procedure of the reference alternative method involves the following steps:

1. Choose an alternative which is dominated by all criteria. Redefine the criteria based on the alternative which was assigned as the reference point.

Principally, any alternative that is relative to all criteria can be chosen as a reference alternative. In practice, we always choose the alternative which has more significant sense as a reference point because it will make the pairwise comparison easier. It is not necessary to choose the "most important" or the "most preferable" alternative as a reference point. In fact, sometimes no alternative exists there which is dominant over all of the other alternatives against all of the criteria. Even though the alternative exists, it can be identified only when the pairwise comparisons are processed and the weight of alternatives against every criterion are derived. Therefore, it may be required to choose the alternative with maximum weight as a reference point. The analysis can also be applied to the alternative with minimum weight.

2. After adjusting the definition of the criteria, make pairwise comparisons of the criteria with respect to the goal and derive the weights of the criteria by the eigenvector method. The weights should be normalized.

The normalization of the weights of the criteria is referred to as the standard process which makes the overall weight of the reference alternative unique. It is different from the normalization in distribution pattern of the AHP.

3. Make pairwise comparisons of alternatives with respect to each criterion and derive the priorities of the alternatives by the eigenvector method. Instead of the normalization procedure, always assign 1 to the weight value of the reference alternative.

It is very important to note that the weight value of the reference alternative is assigned to 1 because the criteria has been defined based on the alternative which was assigned as the reference point. In fact, the weight of the reference alternative is given prior to the pairwise comparisons of criteria and alternatives.

4. Calculate the overall priorities of alternatives with respect to the goal by the following formulation:

$$W_i = \sum_{j=1}^{m} C_j W_{ij}$$

Note that the value of the overall weight of the reference altemative is always equal to 1, i. e. $e_i = 1$ and the sum of overall weights of all the alternatives is greater than 1. It is not necessary to normalize the overall weights even though it may be required for some applications. The normalized overall weights of alternatives have a different meaning from the normalized weights by the distribution pattern of the AHP.

The reference point method

If the reference alternative which meets the two condtions mentioned above can not be found, one should apply the reference point method. The reference point method uses a fixed scale or value of criterion as the reference point to derive weights of criteria instead of an alternative as the reference point. Here is the procedure of the reference point method.

1. Choose an appropriate scale or value of each criterion as the reference point according to alternatives.

The choice of a scale or value as the reference point should match the range of measurement of the alternative under each criterion. For example, $10,000 could be chosen as a reference point of the criterion of cost for a car evaluation, 70 degrees can be chosen as a reference point of the criterion of weather for the evaluation of a living environment.

2. Derive the normalized weights of criteria by pairwise comparison and eigenvector process.

When the pairwise comparisons are made by decision maker's judgments, it is important to keep the meanings of criteria according to the reference points in mind

3. Divide the range of measurement of each criterion into several scale values which must include the reference point. Derive the weights of the scales with respect to each criterion by pairwise comparisons and the eigenvector process. The weights of the reference point against each criterion are always 1.

Dividing scale values of each criterion can be based on an objective physical measurement or a subjective standard measurement. It should avoid an excess of the number of scale values over nine in order to make an acceptable number of pairwise comparisons. This step is necessary because the value of the alternative in question is based on the subjective standard or the objective physical measurement and does not always match the weight value with respect to the criterion in the meanings of the ratio scale or cardnal scale. Therefore, step 3, in fact is the rescale process of the measurement of alternatives.

4. According to the scales of a criterion in step 3 and the measurement of an alternative, assign a weight to the alternative with respect to the criterion. Derive the overall weight of each alternative.

Sometimes the value of the physical measurement of an alternative on a criterion is between two adjoining scales. The weight of the alternative may be assigned to the value which is most closed scale to the value of the alternative or can calculated simply by the procedure of the linear inserting value.

The inconsistency of the pairwise comparisons and adjustment of weights of criteria

The weights of criteria depend on the choice of a reference point. If the judgment matrices of the pairwise comparisons of alternatives with respect to every criterion are perfectly consistent, the relative overall priorities (the normalized weights) of the alternatives will be the same even though the weights of criteria may be different under different reference points. Otherwise, if inconsistent matrices exist there, the relative overall priorities of alternatives will be different under different reference points. When the reference pattern is applied, the effect of the inconsistency of the pairwise comparisons on the decision is more serious than if the distribution pattern is applied. In the case that higher inconsistency occurs, an adjusting procedure should be used to derive reasonable weights of criteria. Another article by the author has an elaborated exploration on the topic.

An example of the reference pattern of the AHP

Below is illustrated how the reference pattern is applied in the faculty hiring example in section III. We denote the four applicants by A_1, A_2, A_3 and A_4, respectively. Suppose that A_1 has been chosen as the reference alternative. To weight the criteria by making pairwise comparisons of criteria, we answer the question: with respect to the evaluation goal, which one is more important, A_1's teaching or research, and to what degree? Assuming the following matrix is given:

$$\begin{vmatrix} 1 & 2 \\ 1/2 & 1 \end{vmatrix}$$

From the matrix, the weights of 0.67 for teaching and 0.33 for research can be derived. With respect to the criterion of teaching, the comparisons of alternatives is made as the following matrix:

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 1/3 & 1 & 2 & 3 \\ 1/5 & 1/2 & 1 & 1/2 \\ 1/4 & 1/3 & 2 & 1 \end{vmatrix}$$

The weights of four applicants against the criterion of teaching can be computed as (1.000, 0.443, 0.172, 0.237). With respect to the criterion of research, the comparisons of alternatives is given as the following matrix:

$$\begin{vmatrix} 1 & 1/3 & 1/7 & 1/5 \\ 3 & 1 & 1/3 & 1/2 \\ 7 & 3 & 1 & 2 \\ 5 & 2 & 1/2 & 1 \end{vmatrix}$$

The weights of four applicants against the criterion of research can be computed as (1.000, 2.700, 8.183, 4.800). Therefore, the overall rankings of the four applicants with respect to the goal are (1.000, 1.195, 2.840, 1.756). Thus, the best applicants is A_3.

Rank Reversal and two Patterns of the AHP

The phenomenon of rank reversal is the most controversial aspect of the AHP. Some authors subjectively assert that the phenomenon of rank reversal is a flaw of the AHP and the rankings provided by the AHP are arbitrary. We disagree with them.

Let us take a moment to look at the first example of rank reversal by Belton and Gear[1]. In their example, three alternatives A_1, A_2 and A_3 are compared against three criteria C_1, C_2, and C_3. The weights of the three alternatives are given as

alternatives	criteria C_1	C_2	C_3
A_1	1	9	8
A_2	9	1	9
A_3	1	1	1

Assuming equal weights on the criteria, the overall priorities of the three alternatives are given as $w_1 = 0.45$, $w_2 = 0.475$ and $w_3 = 0.075$. Then they add a fourth alternative, which is an exact copy of alternative B, i.e. the weights of the four alternatives are given as

	criteria		
alternatives	C_1	C_2	C_3
A_1	1	9	8
A_2	9	1	9
A_3	1	1	1
A_4	9	1	9

The overall priorities of the four alternatives by Belton and Gear are given as $w_1 = .37$, $w_2 = .29$, $w_3 = .06$ and $w_4 = .29$. Then they asserted that in the example the rank reversal phenomenon is inherent in the AHP because the computational scheme is fundamentally flawed.

In fact the example by Belton and Gear is utterly nonsense because they did not point out what type of decision problem it is. If the example is an attribute decision problem, for example, say we want to decide who should be promoted to a higher rank position in a three faculty group according to their working shares of teaching, research and social activity, the overall priority score for each of them are 0.45, 0.474 and 0.075, respectively. If one faculty is added to the group, the overall score for each of these four faculties are 0.37, 0.29, 0.06 and 0.29, respectively. Of course, the rank of the previous three should be reversed because the parts of the attribute weights they share are different from those they previously shared.

Therefore, our conclusion is that the rank reversal phenomenon for multiple attribute decision problems are absolutely reasonable when the AHP is applied.

On the other hand, if the example is a multiple criteria decision problem, like buying a car or admission process, the reference pattern of the AHP should be applied for the example. The overall priority score for each of the four alternatives are given as 0.45, 0.475, 0.75 and 0.475 or as normalized priorities 0.305, 0.322, 0.051 and 0.322, respectively. No rank reversal occurs here. The phenomenon of rank reversal will not occur for the multiple criteria decision problems if the reference pattern of the AHP is applied.

Dyer (1990) provided another example of rank reversal which is shown below:

	Criteria			
alternatives	C_1	C_2	C_3	C_4
A_1	1	9	1	3
A_2	9	1	9	1
A_3	8	1	4	5
A_4	4	1	8	5

Assuming that the four criteria are judged to be equally important, the rankings determined by the AHP for the first three alternatives are 0.320, 0.336 and 0.344, and for the four alternatives are 0.264, 0.243, 0.246 and 0.246, respectively. The alternatives A_1 and A_3 have reversed rankings.

In this example, the results of the alternative rankings are obtained by assuming a multiple attribute decision problem. The rankings are derived by the distribution pattern of the AHP. Rank reversal in this situation is a reasonable phenomenon. If the multiple criteria decision problem is assumed and the reference pattern of the AHP is applied, the rankings of the four alternatives should be 0.200, 0.286, 0.257 and 0.257. There is no rank reversal occurring.

Now one may raise a question: which ranking in these examples is correct? The answer is that the correct ranking depend on the classification of what the decision problem is. In any case the determined rankings would be obtained by the AHP. It is for no reason whatsoever that the rankings by the AHP are arbitrary.

Conclusion

The reasons why the AHP is created, developed and applied in many application fields are that the AHP can be used to deal with a lot of complex and intangible decision factors, it can be used to make decisions by the judgments of the decision maker. The AHP is a reasonable and effective method for measurement and decision making, especially when intangible factors are involved. There are two types of multiple objective decision makings. One is multiple attribute decision making and the other is multiple criteria decision making. One applies the AHP to make decisions, it is important to identify the two types of decision makings and to use the appropriate pattern of the AHP: the distribution pattern and reference pattern in order to obtain a proper result. These two patterns remain the principle of the AHP which includes hierarchy decomposition, 1-9 ratio scale, pairwise comparison and eigenvector procedure to derive the priority. When the reference pattern is used, the normalization process in the hierarchical composition should be eliminated. The phenomenon of rank reversal will not occur by using the reference pattern. When the distribution pattern is applied, phenomenon which represents a change of the distribution of weights of elements in high level to elements in low level, is natural and reasonable. For both patterns, the derived ranking is determinate.

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