A COMPARATIVE STUDY ON AHP AND DEA

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ABSTRACT

Both Analytic Hierarchy Process (AHP) and Data Envelopment Analysis (DEA) aim at making decisions under multiple criteria environments. AHP uses pairwise comparisons and eigenvector weightings, whereas DEA does linear fractional programmings. In this paper, we will point out some structural similarities among them, by comparing the benefit/cost analysis by AHP and DEA. Also, we will discuss on the fixed vs. variable weights in multiple criteria decision making.

1. A GLIMPSE AT DATA ENVELOPMENT ANALYSIS

DEA has been developed by A.Charnes, W.W. Cooper et al. ([1],[2],[3],[4]) since 1978. DEA estimates relative efficiencies of decision making units (DMUs) who have common factors of inputs and outputs. Let the multiple inputs and outputs to a DMU$_j$ $(j=1,\dots,n)$ be $\{x_{ij}: i=1,\dots,m\}$ and $\{y_{rj}: r=1,\dots,s\}$ respectively. We assume that we have $\{x_{ij}\}$ and $\{y_{rj}\}$ in the form of observations or in the form of theoretically prescribed values and their values are positive. Also, we assume that the data are normalized so that they satisfy

$$\sum_{j=1}^{n} x_{ij}=1 \quad (i=1,\dots,m) \tag{1.1}$$

and

$$\sum_{j=1}^{n} y_{rj}=1 \quad (r=1,\dots,s). \tag{1.2}$$

This assumption is laid for comparative study's sake and does not influence any essential features of DEA.
From the efficiency's point of view, a DMU with big outputs relative to small inputs are preferable. We define the relative efficiency of a DMU j_0 by solving the following linear fractional programming:

[FP(j_0)]

$$\max_{u,v} h_{j_0} = \left(\sum_{r=1}^{s} u_r y_{rj_0}\right) / \left(\sum_{i=1}^{m} v_i x_{ij_0}\right) \tag{1.3}$$

subject to

$$(\sum_{r=1}^{s} u_r y_{rj})/(\sum_{i=1}^{m} v_i x_{ij}) \leq 1 \quad (j=1,\ldots,n) \qquad (1.4)$$

$$u_r > 0 \quad (r=1,\ldots,s) \qquad (1.5)$$

and

$$v_i > 0 \quad (i=1,\ldots,m). \qquad (1.6)$$

u_r and v_i are the weights to the r-th output y_r and to the i-th input x_{ij}, respectively. We define the efficiency of a DMU to be the ratio of weighted sum of output values vs. weighted sum of input values. [FP(j_0)] maximizes the ratio associated with the DMU j_0, keeping the ratios of every DMUs, including DMU j_0, not greater than 1.

Let the optimal solution to [FP(j_0)] be u^*, v^* and $h^*_{j_0}$. These values vary from one DMU to another.

[Definition 1]

If $h^*_{j_0}=1$ then the DMU j_0 is <u>DEA-efficient</u>. Otherwise, if $h^*_{j_0}<1$ then the DMU j_0 is <u>DEA-inefficient</u>.

Actually this definition means the following ([5]):

(i) Output Orientation:A DMU is inefficient if it is possible to augment any output without increasing any input and without decreasing any other output.

(ii)Input Orientation:A DMU is inefficient if it is possible to decrease any input without augmenting any other input and without decreasing any output.

A DMU will be characterized as efficient if, and only if, neither (i) nor (ii) obtains.

For an inefficient DMU, it is very important to find out other DMUs which drive the DMU into inefficiency.

[Definition 2]

The <u>efficient frontier</u> to a DMU j_0 is the set of DMUs:

$$E(j_0)=\{j:(\sum_r u^*_r y_{rj})/(\sum_i v^*_i x_{ij})=1,\ j=1,\ldots,n\}, \qquad (1.7)$$

where u^* and v^* are the optimal solutions to [FP(j_0)].

2. BENEFIT/COST ANALYSIS BY AHP

Benefit/cost analysis of AHP consists of two processes, namely benefit process and cost process ([6]). We estimate the benefit priority and the cost priority separately by AHP. Then their ratio gives the relative efficiency of the alternative objects. In this section, first, we consider the b/c analysis in the case of three level perfect hierarchy structure and then show that general cases can be reduced to the three level case.

2.1 Three level perfect graph case

We will deal with three level hierarchy structure as depicted in Fig.1. We call a graph of the structure a <u>perfect hierarchy graph</u> if a node in any level is connected to every nodes in the succeeding level by an arc and is not connected directly to any nodes beyond the succeeding level.

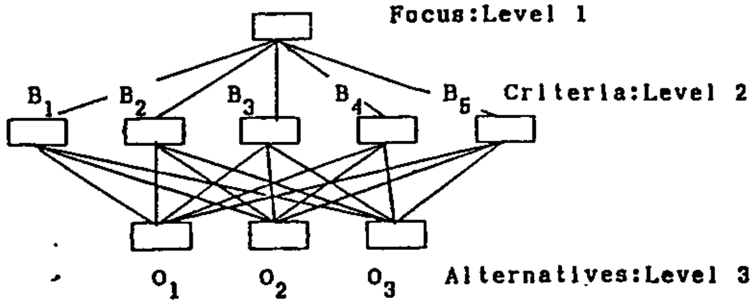

Figure 1. Three level perfect hierarchy graph.

We assume that we have s kinds of benefit criteria $(B_1,\dots,B_s)$ in Level 2 and n kinds of alternative objects $(O_1,\dots,O_n)$ in Level 3. Let y_{rj} be the priority of the object O_j associated with the criteria B_r and U_r be the priority of the criterion B_r. Then, the overall benefit of the object O_j is given by

$$\Sigma_{r=1}^{s} U_r y_{rj}. \qquad (j=1,\dots,n) \qquad (2.1)$$

Here, y_{rj} and U_r satisfy

$$\Sigma_{j=1}^{n} y_{rj}=1 \qquad (r=1,\dots,s) \qquad (2.2)$$

and

$$\Sigma_{r=1}^{s} U_r=1. \qquad (2.3)$$

Similarly, we assume that we have a perfect hierarchy <u>cost</u> structure with m cost criteria $(C_1,\dots,C_m)$. Let x_{ij} be the priority of the object O_j with respect to C_i and V_i be the priority of C_i. They satisfy

$$\Sigma_{j=1}^{n} x_{ij}=1 \qquad (i=1,\dots,m) \qquad (2.4)$$

and

$$\Sigma_{i=1}^{m} V_i=1. \qquad (2.5)$$

Then, the overall cost of O_j is given by

$$\sum_{i=1}^{m} V_i x_{ij}. \qquad (j=1,\dots,n) \qquad (2.6)$$

The benefit/cost priority of the object O_j is evaluated by

$$H_j=(\sum_r U_r y_{yj})/(\sum_i V_i x_{ij}). \qquad (2.7)$$

We notice that in AHP all the elements of x, y, U and V are estimated by the processes of pairwise comparisons and eigenvector weightings or by some other empirical or theoretical evaluations.

2.2 General cases

For a general multi-level structure case, we will reduce it to a three level problem by choosing a <u>key</u> level between focus and alternatives and by aggregating the levels between them as depicted in Fig. 2. If some arcs bypass the key level(Level 2), we will introduce additional nodes in the level so that any path connecting the Level 1 node (focus) to a Level 3 node (alternatives) should meet a node in Level 2.

Also, we will introduce additional dummy arcs with very small x or y values to make the three level structure "perfect", if necessary. It is easy to see that we can calculate the x, y, U and V-values corresponding to the aggregated three level structure from the original values.

Thus, general multi-level cases can be reduced to a three level perfect case by selecting a key level deliberately which usually exists in AHP.

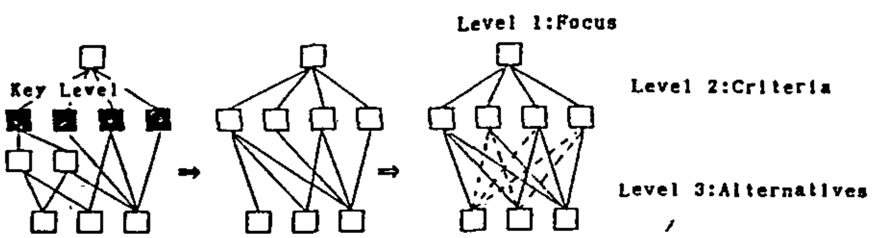

Figure 2. Reduction of general case to three level perfect graph.

3. EFFICIENCIES IN AHP AND DEA

Discussions in Sections 1 and 2 show the structural similarity among b/c analyses by AHP and DEA. Differences exist in the way they estimate x, y, u, v, U and V values.

3.1 Input x and output y

DEA uses available numerical data for input x and output y, while AHP creates them by the processes of pairwise comparisons and eigenvector weightings. Originally, DEA aims at evaluating relative efficiencies of DMUs in the environments where numerical or theoretically prescribed data exist. On the other hand, AHP works in the world where only subjective or psychological factors are prevailing in making decisions. Although both methods stem from extremely different motivations, they exhibit a certain similarity in the presence of data i.e. input x and output y and the ratio scale of efficiency evaluations. They can trade their inputs and outputs.

AHP could be benefited by using the same numerical data with DEA. DEA could expand its world by incorporating qualitative factors that AHP has exposed for the first time.

3.2 Weights

DEA determines the weights u and v by solving the fractional programming [FP(j_0)] corresponding to the decision making unit DMU j_0. Hence, the weights differ from one DMU to another. We will call this kind of weights as <u>variable weights</u>. The weights are determined in such a way that they should be most favorable to the DMU concerned. AHP uses pairwise comparisons and eigenvector weightings in determining the weighs U and V of the key level criteria. The values are common to all alternative objects. We will call this kind of weights as <u>fixed</u>.

3.3 Efficiency

The AHP-efficiency H_j of an object O_j is given by the formula (2.7). The DEA-efficiency of a DMU j_0 is the optimal objective function value to [FP(j_0)]:

$$h^*_{j_0}=(\Sigma\ u^*_r y_{rj})/(\Sigma\ v^*_i x_{ij_0})$$

where u^* and v^* are the optimal solution to [FP(j_0)].

For any AHP-(U,V), let

$$p=\max_j(\Sigma_r\ U_r y_{rj})/(\Sigma_i\ V_i x_{ij}) \qquad (3.1)$$

and $u_r=U_r/p$ (r=1,...,s) and $v_i=V_i$ (i=1,...,m).

Then, (u,v) is feasible to [FP(j_0)].

Conversely, for any DEA-feasible solution (u,v), let $T=\Sigma u_r$ and $S=\Sigma v_i$ and define $U_r=u_r/T$ (r=1,...,s) and $V_i=v_i/S$ (i=1,...,m).
Then (U,V) is an AHP-feasible priority.

Since both transformations are scalings, they have the same priority relations in the b/c analysis.

3.4 Several propositions

The above discussions lead us to several propositions. Throughout this sub section we assume x and y to be constant.

[Proposition 1]

For any AHP weight (U,V), there exists a DMU j_0 that has the transformed (u,v) as the optimal solution to [FP(j_0)]. Indeed, j_0 is the DMU that gives the maximum value to (3.1).

[Proposition 2]

DEA is the most generous one among the multiple criteria methods for evaluating the efficiency of DMUs by ratio scale in the sense that an efficient DMU under the latter criteria has a corresponding DEA optimal weight (u,v) which makes the DMU be DEA-efficient.

[Proposition 3]

A DEA-inefficient DMU is also AHP-inefficient by any weighting of the criteria. Moreover, a DEA-inefficient DMU is inefficient under any fixed weight multiple criteria benefit/cost analysis.

4. CONCLUDING REMARKS

Both AHP and DEA have turned out to give strong impulses to the multiple criteria decision making community, although their origins and motivations are quite different.

In this short report, we pointed out structural similarities among them in case of the b/c analysis and suggested their potential trades.

In short, AHP could be more objective by incorporating the DEA-efficiency. AHP can exclude essentially inefficient objects by using DEA-inefficiency. Conversely, DEA could be more subjectively oriented by incorporating some features of AHP. For example, by adding such constraints as $u_1 \geqq u_2$ or $3v_1 \leqq v_2$ to [FP(j_0)], DEA would become more intensive in judging the efficiency of the DMU concerned.

Although we have concerned mainly with the comparative study on the b/c analyses of AHP and DEA, it should be noted that the usual AHP could be regarded as a special case of the b/c AHP where the cost factor has only one criterion with equal weight to each object of alternatives. Hence, the Propositions 1 to 3 remain valid in the latter case where the corresponding [FP(j_0)] of DEA reduces to a linear programming.

REFERENCES

[1]Banker,R.D. "Estimating Most Productive Scale Size Using Data Envelopment Analysis," European J. Oper. Res., (1984),35-44.

[2]Banker,R.D.,A.Charnes and W.W.Cooper,"Some Models for Estimating Technical and Scale Inefficiencies," Management Sci.,(1984),1078-1092.

[3]Charnes,A.,W.W.Cooper,B.Golany and L.Seiford,"Foundation of Data Envelopment Analysis for Pareto-Koopmans Efficient Empirical Production Functions," J. Econometrics, (1985),91-107.

[4]Charnes,A.,W.W.Cooper and E.Rhodes,"Measuring the Efficiency of Decision Making Units," European J. of Oper. Res., (1978),429-444.

[5]Charnes,A.,W.W.Cooper and E.Rhodes,"Evaluating Program and Managerial Efficiency," Management Sci.,(1981),668-697.

[6]Saaty,T.L.<u>The Analytic Hierarchy Process</u>, McGraw-Hill, 1980.