TWO TECHNICAL NOTES ON THE AHP BASED ON GEOMETRIC MEAN METHOD

Kaoru Tone
Graduate School of Policy Science
Saitama University, Urawa Saitama 338, Japan
tone@poli-sci.saitama-u.ac.jp

Abstract

This paper describes two technical subjects of the Analytic Hierarchy Process. The first one relates with group decision making. When we estimate the relative distance between cities or the relative area of figures by the AHP, it is often observed that a group decision usually outperforms an individual one. This note addresses this phenomena and shows that the accuracy of estimates is improved in approximate proportion to the square root of the number of individuals in the group.

The second one deals with the method for estimating the relative weight of alternatives when some entries of the pairwise comparisons matrix are missing, based on the geometric mean method.

1 Group vs. Individual Decision Making in the AHP

1.1 Introduction

Saaty's AHP [3] is now being widely used for decision making purposes. One of the important factors in the AHP is the pairwise comparison of alternatives (or criteria) in the problem. Usually, there are two kinds of pairwise comparison, i.e., by an individual and by a group. In classroom experiments for measuring the relative distance between cities on a map or the relative area of figures, the author has often observed the result that a group decision outperforms an individual one in accuracy. This section shows that theoretically, the accuracy of estimates is improved in approximate proportion to the square root of the number of individuals in the group, if the members are unbiased and homogeneous.

1.2 Eigenvalue Method and Geometric Mean Method

Let the pairwise comparison matrix be

$$A = [a_{ij}], \tag{1}$$

where $a_{ii} = 1 \quad (i = 1, \ldots, n), \quad a_{ij} = 1/a_{ji} \quad (\forall \ (i,j))$, and $\quad a_{ij} > 0 \quad (\forall \ (i,j))$. There are two methods for estimating the relative weight of the alternatives.

1. Eigenvalue Method:
   This method solves the principal eigenvalue of A and its eigenvector. Let the eigenvalue and the eigenvector be λ_{max} and v, respectively. We assume the eigenvector is normalized so that the sum of the elements of v is 1.

2. Geometric Mean Method:
   The geometric mean method (GM) or the logarithmic least squares method works as follows: Minimize, with respect to $g = (g_i) \in R^n$,

$$\sum_{i,j=1}^{n} (\log a_{ij} - \log g_i/g_j)^2 . \tag{2}$$

   It turns out that a GM (LLSM) solution g is given by the geometric mean of elements in each row of A, i.e.,

$$g_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}} . \quad (i = 1, \ldots, n) \tag{3}$$

The vector g is normalized as

$$g_i' \leftarrow g_i / \sum_{j=1}^{n} g_j \quad i = 1, 2, \ldots, n. \tag{4}$$

The two approaches give almost the same weights v and g', if the matrix A is nearly consistent. (See Golden and Wang [1] and Tone [7]. Also, see Takeda [6] for further extensions of the geometric mean method.) So, hereafter, we will deal with the geometric mean method (GM), since GM is easier for analyzing the above mentioned subjects.

1.3 Perturbation of Pairwise Comparison Matrix

We assume that the true weight vector $w = (w_i)$ exists. The (i, j) element of the ideal comparison matrix is expressed as

$$\frac{w_i}{w_j}. \tag{5}$$

The estimated comparison value a_{ij} is an approximation to w_i/w_j and let relate with it by

$$a_{ij} = \frac{w_i}{w_j} e^{\varepsilon_{ij}}, \tag{6}$$

where ε_{ij} is a random variable representing the deviation from the true value. We assume that ε_{ij} has the mean zero and the variance σ_{ij}^2.

The above setting matches with the exponential scoring of pairwise comparisons. If ε_{ij} is small, then we have

$$e^{\varepsilon_{ij}} = 1 + \varepsilon_{ij} + O(\varepsilon_{ij}^2). \tag{7}$$

Therefore, (6) can be written as

$$a_{ij} = \frac{w_i}{w_j} \left(1 + \varepsilon_{ij} + O(\varepsilon_{ij}^2)\right). \tag{8}$$

Thus, ε_{ij} can be interpreted as a relative error to w_i/w_j.

1.4 Effect of Perturbation on Weight

If we calculate the weight by GM, using the perturbed matrix (6), we have

$$g_i = \frac{w_i}{\sqrt[n]{\prod_{j=1}^{n} w_j}} e^{\left(\sum_{j=1}^{n} \varepsilon_{ij}\right)/n}, \quad (i = 1, \ldots, n) \tag{9}$$

where $\varepsilon_{ii} = 0$ $(\forall i)$. By normalizing g, we have the estimated weight

$$g_i' = \frac{w_i e^{\left(\sum_{j=1}^{n} \varepsilon_{ij}\right)/n}}{\sum_{j=1}^{n} w_j e^{\left(\sum_{k=1}^{n} \varepsilon_{jk}\right)/n}}. \quad (i = 1, \ldots, n) \tag{10}$$

Under the small ε_{ij} hypothesis, g_i' can be approximated by

$$\begin{aligned} g_i' &= w_i \Bigg[1 - \frac{1}{n} \Big\{ \sum_{k=1}^{i-1} (w_k - w_i + 1) \varepsilon_{ki} + \sum_{k=i+1}^{n} (w_i - w_k - 1) \varepsilon_{ik} \\ &\quad + \sum_{j<k, (j,k \neq i)} (w_j - w_k) \varepsilon_{jk} \Big\} + \frac{1}{n^2} O\left(\varepsilon_{jk}^2\right) \Bigg]. \end{aligned} \tag{11}$$

Let us observe the first order term in ε in (11), which can be regarded as the relative error of the estimated g_i' from w_i under the small ε hypothesis:

$$\begin{aligned}\delta_i &= -\frac{1}{n}\left[\sum_{k=1}^{i-1}(w_k - w_i + 1)\,\varepsilon_{ki} + \sum_{k=i+1}^{n}(w_i - w_k - 1)\,\varepsilon_{ik}\right. \\ &\quad \left. + \sum_{j<k,(j,k\neq i)}(w_j - w_k)\,\varepsilon_{jk}\right]. \quad (i = 1,\ldots,n)\end{aligned} \tag{12}$$

If we assume that ε_{jk} distributes independently with the mean 0 and the variance σ^2, then δ_i is a random variable with the mean 0 and the variance V_i as

$$V_i = \frac{\sigma^2}{n}\left(\sum_{j=1}^{n} w_j^2 - 2w_i + 1\right). \tag{13}$$

(See Appendix for derivation). Since,

$$\sum_{j=1}^{n} w_j^2 - 2w_i + 1 = \sum_{j=1,j\neq i}^{n} w_j^2 + (1-w_i)^2 \leq 2, \tag{14}$$

we have:

Theorem 1 *The estimated g_i' has a relative error approximately proportional to $\sigma/\sqrt{n}$.*

1.5 Effect of Group Decision on Weight

We observe the case where m individuals do the pairwise comparisons independently and make the matrix A by their geometric mean. Thus, we have

$$a_{ij} = \frac{w_i}{w_j} e^{\left(\sum_{k=1}^{m}\varepsilon_{ijk}\right)/m}, \tag{15}$$

where ε_{ijk} is a random variable corresponding to the error term of the k-th individual. The group decision weight can be determined by the row-wise geometric mean of A:

$$\bar{g}_i = \frac{w_i}{\sqrt[n]{\prod_{j=1}^{n} w_j}} e^{\left(\sum_{j=1}^{n}\sum_{k=1}^{m}\varepsilon_{ijk}\right)/nm}. \quad (i = 1,\ldots,n : \varepsilon_{iik} = 0) \tag{16}$$

In the same way as (11), we can approximate $\bar{g}_i$ by

$$\begin{aligned}\bar{g}_i &= w_i\left[1 - \frac{1}{nm}\sum_{k=1}^{m}\left\{\sum_{h=1}^{i-1}(w_h - w_i + 1)\,\varepsilon_{hik}\right.\right. \\ &\quad \left. + \sum_{h=i+1}^{n}(w_i - w_h - 1)\,\varepsilon_{ihk} + \sum_{j<h,(j,h\neq i)}(w_j - w_h)\,\varepsilon_{jhk}\right\} \\ &\quad \left. + \frac{1}{(nm)^2}O\left(\varepsilon_{jhk}^2\right)\right].\end{aligned} \tag{17}$$

Here again, we assume that $\varepsilon_{jhk}(\forall(jhk))$ subjects to a distribution with the mean 0 and the variance σ^2,i.e., unbiased and homogeneous. Under the small ε hypothesis, the first order terms of ε in (17) correspond to the relative error of $\bar{g}_i$ to w_i, whose mean is 0 and variance is:

$$\bar{V}_i = \frac{\sigma^2}{nm}\left(\sum_{j=1}^{n} w_j^2 - 2w_i + 1\right). \tag{18}$$

By comparing (18) with the individual case (13) discussed in the preceding subsection, we have:

Theorem 2 *The group decision by m individuals reduces the error of the estimated weight by the factor $1/\sqrt{m}$, if the members of the group are unbiased and homogeneous.*

1.6 Concluding Remarks

This section discussed the relative error of judgements by the geometric mean method in terms of the relative error in the pairwise comparisons and evaluated those of individual and group decisions. As a consequence, we showed that the group decision improves the accuracy of estimated weight in proportion to the square root of the number of individuals in the group, if the members are 'unbiased and homogeneous'. On the 'unbiased' issue, the approximate *Consistency Index* (C.I.) below can be usefully applied.

$$\text{C.I.} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(g'_j/g'_i) - n^2}{n(n-1)}. \tag{19}$$

If a member's C.I. (or the corresponding C.R.) is greater than 0.1, his comparison matrix must be retried or deleted from the group decision. As to the 'homogeneity' issue, Saaty [5] will contribute to a better understanding of the matter.

2 A Logarithmic Least Squares Method for Incomplete Pairwise Comparisons Matrix

2.1 Introduction

One major drawback of the AHP is that at each level in the hierarchy, $n(n-1)/2$ questions must be answered. The number of questions grows very large with n. In addition, for certain pairs (i,j), it is very difficult to answer the question "compare i against j". This results in some entries of A being missing. Therefore, methods for estimating the weight of alternatives from the incomplete matrix are requested. Harker [2] solved this problem effectively in the framework of the eigenvalue method. The main purpose of this section addresses the solution to the incomplete matrix problem by the logarithmic least squares principle.

2.2 Logarithmic Least Squares for Incomplete Pairwise Comparisons Matrix

We can define an undirected graph corresponding to the paired comparisons with the vertices $1, 2, \ldots, n$ and with arcs $(i,\ j)$ if i and j are compared directly.

Definition 1 *We call a pairwise comparisons matrix* incomplete, *if*

1. *the corresponding graph is connected and*
2. *is not a perfect graph.*

Let an incomplete matrix be $A = (a_{ij})$. For each vertex i, we define P_i as the set of vertices adjacent to i and N_i as the degree of i , i.e., the number of arcs connected to i. Since the graph is connected, for each i, P_i is nonempty and $N_i \geq 1$. For the missing matrix entries a_{ij}, let us approximate their value by the ratio of the (yet unknown) weights g_i/g_j. For the purpose of obtaining the weight g, we solve the following logarithmic least squares problem:

$$\textit{minimize} \qquad \sum_{i,j} (\log a_{ij} - \log g_i + \log g_j)^2 \tag{20}$$

$$= \sum_{i=1}^{n} \left[\sum_{j \in P_i} (\log a_{ij} - \log g_i + \log g_j)^2 \right] \tag{21}$$

The problem results in a set of linear equations in $(\log g_j)$ as

$$N_i \log g_i - \sum_{j \in P_i} \log g_j = \sum_{j \in P_i} \log a_{ij}. \quad (i = 1, 2, \ldots, n) \tag{22}$$

Example 1
The matrix below has entries (1,3), (2,4) and (3,4) missing.

$$A = \begin{pmatrix} 1 & a_{12} & g_1/g_3 & a_{14} \\ a_{21} & 1 & a_{23} & g_2/g_4 \\ g_3/g_1 & a_{32} & 1 & g_3/g_4 \\ a_{41} & g_4/g_2 & g_4/g_3 & 1 \end{pmatrix}.$$

The corresponding linear equations are

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \log g_1 \\ \log g_2 \\ \log g_3 \\ \log g_4 \end{pmatrix} = \begin{pmatrix} \log a_{12}a_{14} \\ \log a_{21}a_{23} \\ \log a_{32} \\ \log a_{41} \end{pmatrix}.$$

The rule for constructing the coefficient matrix of the linear equations is:
(1) Put -1 on the compared entries and 0 on the missing ones, and
(2) on the diagonal entries, put the number of -1s on the row.

Let the coefficient matrix be D. Then, we have

Theorem 3 *The rank of the matrix D is $n-1$, if and only if the graph of the pairwise comparisons is connected.*

Proof. First, we show the 'if-part'. Since the sum of n row vectors of D is zero, the rank of D is less than $n-1$. Let D_{n-1} be the left upper $(n-1)\times(n-1)$ matrix of D. For a vector $x = (x_j) \in R^{n-1}$, the quadratic form associated with D_{n-1} is:

$$Q = x^T D_{n-1} x = \sum_{i=1}^{n-1} N_i x_i^2 + 2 \sum_{1\le i<j\le n-1} d_{ij} x_i x_j \tag{23}$$

$$= \sum_{1\le i<j\le n-1,\ d_{ij}=-1} (x_i - x_j)^2 + \sum_{i=1}^{n-1} \left(N_i + \sum_{j=1,\neq i}^{n-1} d_{ij} \right) x_i^2 \tag{24}$$

We observe the case $Q = 0$.
(i) If the first term $(x_i - x_j)^2$ on the right-hand side of (24) is not vacant, then, for each i, we have, under the condition $Q = 0$,

$$x_i = x_j. \quad (\forall j \in P_i) \tag{25}$$

Furthermore, at least one of x_i and (x_j) $(j \in P_i)$ has the term x_i^2 or x_j^2 in the second term on the right-hand side of (24), since otherwise the vertices x_i and (x_j) $(j \in P_i)$ are disconnected to the remaining ones and this contradicts the connected graph hypothesis. Thus, we have, for each i in the first term,

$$x_i = x_j \quad (\forall j \in P_i) = 0. \tag{26}$$

(ii) For x_k not included in the first term, we have x_k^2 in the second term. Hence $x_k = 0$.

Thus, if $Q = 0$, then $x = 0$. Therefore, all the eigenvalue of D_{n-1} is positive and the rank of D_{n-1} is $n-1$.

The 'only-if' part can be demonstrated as follows. Suppose the graph is disconnected. Then, the matrix D can be decomposed, after rearrangement, into

$$D = \begin{pmatrix} D_1 & O \\ O & D_2 \end{pmatrix}, \tag{27}$$

where $D_1 \in R^{n_1\times n_1}$, $D_2 \in R^{(n-n_1)\times(n-n_1)}$ and $n_1 > 0$. The ranks of D_1 and D_2 are less than or equal to $n_1 - 1$ and $n - n_1 - 1$, respectively. Hence, the rank of D must be less than or equal to $n-2$, since the rank is the maximum number of linearly independent columns (or rows) of the matrix. □

2.3 A Geometric Mean Method for Incomplete Pairwise Comparisons

Based on the preceding theorem, a geometric mean method for incomplete pairwise comparisons goes as follows:

1. Let any one of $(\log g_j)$ $(j = 1, \ldots, n)$ be zero and solve the equations (22) in remaining $(n-1)$ unknowns.
2. Obtain the weight g_j from $\log g_j$ for $(j = 1, \ldots, n)$.
3. Normalize (g_j) so that
$$g'_j = \frac{g_j}{\sum_{k=1}^{n} g_k}. \quad (j = 1, \ldots, n) \tag{28}$$

Example 2
Let an incomplete pairwise comparisons matrix A be as below, where the symbol $-$ stands for uncompared entries:

$$A = \begin{pmatrix} 1 & - & 3 & 2 \\ - & 1 & 9 & 6 \\ 1/3 & 1/9 & 1 & - \\ 1/2 & 1/6 & - & 1 \end{pmatrix}.$$

The corresponding linear equations are

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \log g_1 \\ \log g_2 \\ \log g_3 \\ \log g_4 \end{pmatrix} = \begin{pmatrix} \log 3 \times 2 \\ \log 9 \times 6 \\ \log(1/3) \times (1/9) \\ \log(1/2) \times (1/6) \end{pmatrix}.$$

We assume $\log g_4 = 0$ and solve the system for $\log g_j$ $(j = 1, 2, 3)$ which gives

$$\log g_1 = \log 2, \; \log g_2 = \log 6, \; \log g_3 = \log(2/3), \; \log g_4 = 0.$$

Thus, we obtain the normalized weight

$$g' = (0.207, \; 0.621, \; 0.069, \; 0.103). \tag{29}$$

2.4 Concluding Remarks

This section dealt with the incomplete pairwise comparisons in the AHP within the framework of the logarithmic least squares method. It is easy to see that the weight thus obtained has perfect consistency, if the estimates in the compared entries are consistent. A measure of consistency can be defined by

$$G = \frac{\sum_{i=1}^{n} \left(\sum_{j \in P_i} a_{ij} g_j / g_i - N_i \right)}{\sum_{i=1}^{n} N_i}, \tag{30}$$

which is an average deviation of the compared estimate a_{ij} from g_i/g_j. Obviously, G is nonnegative and equal to zero if and only if the estimate a_{ij} satisfies

$$a_{ij} = \frac{g_i}{g_j} \quad (\forall (i, j)). \tag{31}$$

However, it should be noted that, if, in the most incomplete case, the graph is a spanning tree, the calculated weight (g_i) is always consistent and hence $G = 0$. This observation suggests the need for other indices of accuracy of measurement for incomplete comparisons. This is a future research subject.

Appendix: Derivation of (13)

$$\begin{aligned}
V_i &= \frac{\sigma^2}{n^2}\left[\sum_{k=1}^{i-1}(w_k - w_i + 1)^2 + \sum_{k=i+1}^{n}(w_i - w_k - 1)^2 + \sum_{j<k,(j,k\neq i)}(w_j - w_k)^2\right] \\
&= \frac{\sigma^2}{n^2}\left[\sum_{k=1(k\neq i)}^{n}(w_i - w_k - 1)^2 + \sum_{j<k,(j,k\neq i)}(w_j - w_k)^2\right] \\
&= \frac{\sigma^2}{n^2}\left[\sum_{j<k}(w_j - w_k)^2 + (n-1) - 2(n-1)w_i + 2\sum_{k=1,\neq i}^{n} w_k\right] \\
&= \frac{\sigma^2}{n^2}\left[(n-1)\sum_{j=1}^{n} w_j^2 - 2\sum_{j<k} w_j w_k - 2nw_i + n + 1\right] \\
&= \frac{\sigma^2}{n^2}\left[n\sum_{j=1}^{n} w_j^2 - \left(\sum_{j=1}^{n} w_j\right)^2 - 2nw_i + n + 1\right] \\
&= \frac{\sigma^2}{n}\left[\sum_{j=1}^{n} w_j^2 - 2w_i + 1\right]. \square
\end{aligned}$$

References

[1] Golden, B.L. and Q. Wang,"An Alternate Measure of Consistency", in B.L.Golden, E.A.Wasil and P.T.Harker (Eds), *The Analytic Hierarchy Process:Application and Studies*, pp.68-81, Springer, (1989).

[2] Harker, P.T., "Incomplete Pairwise Comparisons in the Analytic Hierarchy Process", *Mathematical Modelling*, **9**, 837-848 (1987).

[3] Saaty, T.L.,*The Analytic Hierarchy Process*, McGraw-Hill, (1980).

[4] Saaty, T.L., "Eigenvalue and Logarithmic Least Squares", *European Journal of Operations research*, **48**, 156-160 (1990).

[5] Saaty, T.L., "Group Decision Making and the AHP",in B.L.Golden, E.A.Wasil and P.T.Harker (Eds), *The Analytic Hierarchy Process:Application and Studies*, pp.59-67, Springer, (1989).

[6] Takeda, E., "A Note on Consistent Adjustments of Pairwise Comparison Judgments ", *Mathematical and Computer Modelling*, **17**, 7, 29-35 (1993).

[7] Tone,K.,"A Practical Procedure for Finding Copy-like Alternatives in the AHP", (in Japanese),*Communications of the Operations Research Society of Japan*, **36**, 4, 190-194, 1991.