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AGGREGATING INDIVIDUAL JUDGMENTS AND PRIORITIES WITH THE ANALYTIC HIERARCHY PROCESS

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Abstract

The Analytic Hierarchy Process (AHP) is often used in group settings where group members engage in discussion and arrive at consensus judgments. Alternatively, each member of the group can make individual judgments. This paper considers alternative ways to aggregate individual judgments. We show why geometric, rather than arithmetic means be used. We also consider that the method used to aggregate priorities or judgments depends on the nature of the group.

Introduction

Saaty's (1980) Analytic Hierarchy Process (AHP) is one of the most popular and powerful techniques for decision making in use today. AHP is generally used to derive priorities based on sets of pairwise comparisons. The AHP is built on the philosophical understanding of the fundamentals of how the human mind works, hence it should not create a paradox if its mechanics are meaningfully applied meaningfully to the problem at hand. There are several possible ways to aggregate information when more than one (perhaps many) individuals participate in a decision process, including: (1) aggregating the individual judgments for each set of pairwise comparisons into an 'aggregate hierarchy'; (2) synthesizing each of the individual's hierarchies and aggregating the resulting priorities; (3) aggregating the individual's derived priorities in each node in the hierarchy; and (4) including the decision makers as elements in the hierarchy. We will focus on the first two of these methods here and refer to them as AIJ (aggregating individual judgments) and AIP (aggregating individual priorities) respectively.

Ramanathan and Ganesh (1994) have proposed that a weighted *arithmetic* mean of the individuals' priorities (AIP) be used to combine individual input in order to satisfy the Pareto principle of social choice theory. This paper will examine reasons for using geometric rather than arithmetic averages and will consider when the Pareto principle is relevant and under what circumstances AIJ or AIP should be used.

Arithmetic vs. Geometric means

Most people were taught, and have grown up to feel comfortable with the arithmetic mean, or what is commonly referred to as just the mean or average. Grolier's Multimedia Encyclopedia (1993) provides the following definition for the mean or average:

"The mean, or average, of a set of numbers is the sum of all the numbers in the set divided by the total number of elements of the set. Taking as an example the set 50, 60, 65, 70, 75, 80, 80, 85, 92, the mean is 73 (657 divided by 9)."

Grolier's continues however:

" The mean should more precisely be called the arithmetic mean to distinguish it from other, more specialized types of means, such as the geometric, harmonic, and weighted means." The American Heritage Dictionary (1987) provides a similar definition, and goes on to define the geometric mean:

> mean: A number that represents a set of numbers in any of several ways determined by a rule involving all members of the set; average. b. The arithmetic mean

arithmetic mean: The number obtained by dividing the sum of a set of quantities by the number of quantities in the set.

geometric mean: The n th root, usually the positive n th root, of a product of n factors."

Levels of measurement

Stevens' (1968) has categorized the meaning of numbers into the following categories (or what are commonly called scales or levels of measurement):

Nominal
Ordinal
Interval, and
Ratio.

Each category or scale has more meaning than the preceding category. Thus, for example, while interval scale numbers convey *ordinal* meaning as do ordinal numbers, the *intervals* in an interval scale convey meaning while the *intervals* in an ordinal scale do not. Similarly, while ratio scale numbers convey *ordinal* and *interval* meaning as do interval numbers, the *ratios* in a ratio scale have meaning while the *ratios* in an interval scale do not. The types of mathematical operations permitted with numbers is constrained by the level of measurement. For example, adding ordinal scale numbers is mathematically meaningless. Thus, the arithmetic average of ordinal numbers is meaningless --for example the average rating from a scale of (1) poor, (2) fair, (3) good, and (4) excellent is meaningless. Similarly, multiplying or dividing interval level numbers is mathematically meaningless. For example, the coefficient of variation (defined as the variance divided by the mean) is as meaningful as measuring the length of a book with a rubber band -- if the underlying measurements are not on a ratio scale.

The geometric mean requires ratio level data

If we want to take an 'average' of measurements possessing only interval level meaning, we must use the arithmetic average since it is meaningless to multiply interval level numbers. However, if we have ratio level measurements (as we to with AHP) we can calculate *both* an arithmetic average and a geometric average from ratio scale measurements.

Comparing arithmetic and geometric means

The arithmetic mean of (5,5) is the same as the geometric mean, namely 5. However, the arithmetic mean of (1,9) is 5 while the geometric mean is 3. The arithmetic mean of (.1, 9.9) is still 5 whereas the geometric mean is 0.995.
Because of familiarity and the ease of calculation, most of us feel more "comfortable" with the arithmetic mean as a measure of 'central' tendency. However we would become more 'comfortable' with the geometric mean if we used it more, and ease of calculation is far less important given today's computational technologies.

Alternative definitions of arithmetic and geometric means

James & James (1968) provide definitions of arithmetic and geometric means that provide additional insight:

The arithmetic average of two numbers is the middle term in an arithmetic progression of three terms including the two given numbers.

The geometric average of two numbers is the middle term in a geometric progression of three terms including the two given numbers.

Thus, the arithmetic average of 1 and 9 is 5 since **5** is four more than 1 and 9 is four more than **5**. The geometric average of 1 and 9 is 3 since **3** is three times 1 and 9 is three times **3.**

Other examples are shown in Figure 1. The first four examples were chosen so that the sum was held constant at 10, hence the arithmetic average is always 5, and the geometric average varies. The second three examples were chosen such that the product was held constant at 25, hence the geometric average is always 5 while the arithmetic average varies.

First of two numbers	Arithmetic mean	Geometric mean	**Second of two numbers**	Difference between arithmetic mean and first number	Difference between 2nd number and arithmetic mean	Ratio of geometric mean to 1st number	Ratio of 2nd number to geometric mean
5	5	5	5	0	0	1	1
1	5	3	9	4	4	3	3
0.1	5	.995	9.9	4.9	4.9	9.95	9.95
0.01	5	.316	9.99	4.99	4.99	31.607	31.607
1	13	5	25	12	12	5	5
0.1	125.05	5	250	124.95	124.95	50	50
0.01	1250.005	5	2500	1249.995	1249.995	500	500

Figure 1

Why use the geometric mean with AHP?

Aczel and Saaty (1983) have shown that when aggregating the judgments of n individuals where the reciprocal property is assumed even for a single n-tuple, only the geometric mean satisfies the unanimity condition (if all individuals give the same judgment x, that judgment should be the synthesized judgment) and the homogeneity condition (if all individuals judge a ratio t times as large as another ratio, then the synthesized judgment should also be t times as large). Another, perhaps more intuitive reason to use geometric means is that the geometric mean is consistent with the meaning of *both* judgments and priorities in AHP. In particular, Judgments in AHP represent how many times more important (preferable) one factor is than another. Synthesized alternative priorities in AHP are ratio level measures and have meaning such that the ratio of two alternatives' priorities represents how many times more preferable one alternative is than the other. For example, in the case of two judgments (or priorities), the use of the geometric mean is consistent with the definition: "the middle term in a geometric progression of three terms including the two given numbers".

Two alternative geometric aggregation methods with AHP.

Two ways to aggregate information when a group of individuals participate in a decision process are: (1) aggregate the individual judgments for each set of pairwise comparisons into an 'aggregate hierarchy' and synthesize (AIJ), and (2) synthesize each of the individual's hierarchies and aggregate the resulting individual priorities (AIP). Saaty (1994) provides guidance as when to use each of these two methods. Saaty recommends that when the group consists of individual *experts*, all the individuals form the hierarchy but each works out his or her own assessment. Then a geometric mean of the priorities (outcomes) is used (AIP). If, on the other hand, the group consists of *tyros*, then judgments for each comparison should be combined (into one aggregate hierarchy) using the geometric mean. This aggregation of individual judgments is referred to as AIJ.

Aggregating Individual Judgments (AIJ)

When individuals are willing or must relinquish their own preferences (values, objectives) for the good of the organization, they act in concert and pool their judgments in such a way that the group becomes a new 'individual' and behaves like one. There is a synergistic aggregation of individual judgments. Individual identities are lost with every stage of aggregation. We are not concerned with individual priorities. Consequently there is no synthesis for each individual and the Pareto principle is irrelevant.

Since the group becomes a new 'individual' and behaves like one, the reciprocity requirement for the judgments must be satisfied and consequently the geometric mean must be used for reasons given above.

Aggregating Individual Priorities (AIP)

When individuals are each acting in his or her own right, with different value systems, we are concerned about each individual's resulting alternative priorities. An aggregation of each individual's resulting priorities can be computed using either a geometric or arithmetic mean.

1. The aggregation of individual priorities will satisfy the Pareto principle with *either* an arithmetic or geometric average:

   If $a_i \geq b_i, i = 1,2,....n$ then $\dfrac{\sum_{i=1}^{n} a_i}{n} \geq \dfrac{\sum_{i=1}^{n} b_i}{n}$ for an arithmetic mean, and

   $\sqrt[n]{\prod_{i=1}^{n} a_i} \geq \sqrt[n]{\prod_{i=1}^{n} b_i}$ for a geometric mean, provided $a_i \geq 0$ *and* $b_i \geq 0, i = 1,2,....n$.

2) Ramanathan and Ganesh suggest only arithmetic but we maintain either arithmetic or geometric can be used and there are advantages for each.

Advantage for arithmetic – simpler, people are more accustomed to it, and it corresponds to a model with Goal, individuals, criteria, alternatives

If you want to capitalize on the ratio meaning of the priorities, (so that the central tendancy is a central tendancy of ratios rather than differences then the geometric mean should be used . Note, the geometric mean can NOT be used unless one has a ratio scale.

Pathological cases

Ramanathan and Ganesh (1994) have illustrated that, using the AIJ method (which they refer to as the Geometric Mean Method or GMM), it is possible to violate a commonly accepted choice axiom referred to as the Pareto principle, whereby each of the decision makers in a group could, individually prefer one alternative, say A, but a different alternative, say B, has the highest priority when individual judgments are aggregated and the resulting hierarchy synthesized. They recommended that an arithmetic average of individual priorities always be used to aggregate individual judgments. Their example consisted of five criteria and three alternatives, and where the relative importance of the criteria varied as well as the preference for the alternatives with respect to each criterion. We will illustrate the same phenomenon with an even simpler example consisting of only three criteria and two alternatives and where the relative importance of the criteria are equal.

Simplified example

The judgments in this example were selected to show how a violation of the Pareto principle can occur with the judgment aggregation method. The individual and geometric averages of the judgments with respect to the three criteria are shown in Figures 5, 6 and 7 respectively. The judgments are shown in a 'graphical form'. An expanded scale (beyond the traditional 1-9 verbal scale) is used to emphasize the great impact extreme judgments have on the geometric mean. The judgments in Figure 5 are: 1:2, 1:2, and 99:1 for individuals 1,2 and 3 respectively. Individuals 1 and 2 judging A2 to be more preferable than A1, but because Individual 3 judges A1 to be more preferable than A2 by such a large ratio, the geometric average is 2.9:1, A1 preferred to A2.

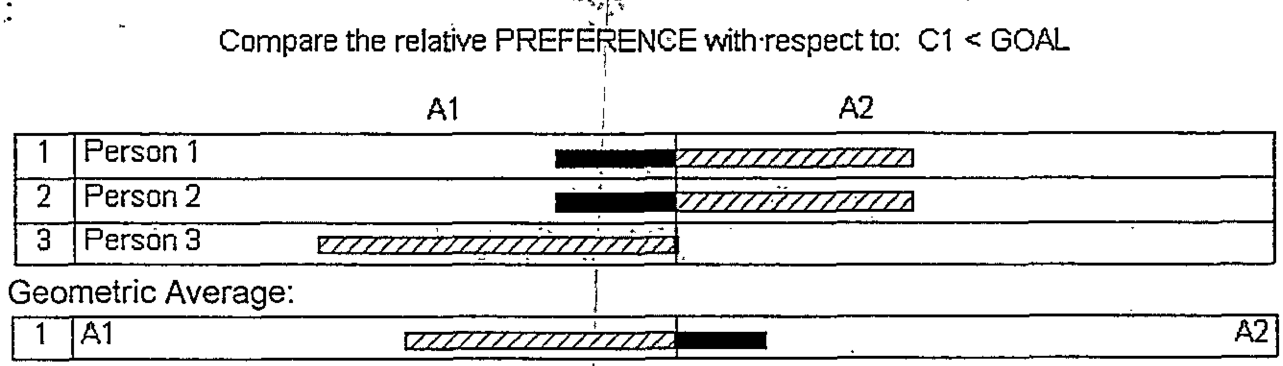

Figure 5 -- Individual judgments with respect to first criterion

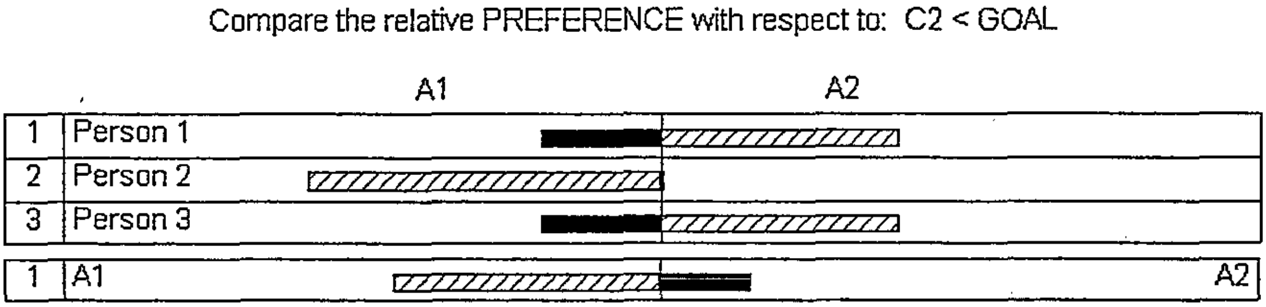

Figure 6 -- Individual judgments with respect to second criterion

The judgments with respect to the second criterion (shown in Figure 6) are similar to those with respect to the first criterion, except now A2 is judged to be more preferable than A1 by Individuals 1 and 3, while Individual 2 judges is the outlier, judging A1 to be much more preferable than A2. Thus far, with respect to the first two criteria, A2 is judged to be more preferable than A1 by two of the three individuals, but because the judgment of the third individual is of opposite direction and of extreme magnitude, the geometric mean is that A1 is preferred to A2. The completion of this pathological example is shown in Figure 7, where each of the three individuals judge A2 to be 9 times more preferable than A1, and the resulting geometric mean is obviously that A2 is 9 times more preferable than A1.

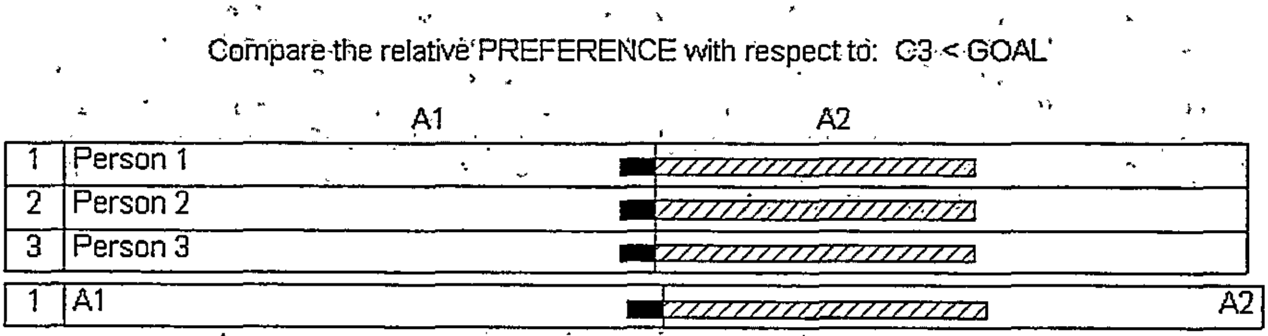

Figure 7 -- Individual judgments with respect to third criterion

If we synthesize each individual's model separately, A2 turns out to be more preferable than A1 (as can be seen by the three rightmost bars in Figures 8 or 9). This is obvious since individual 1 judged A2 to be more preferable than A1 on each of the three criteria. Both individuals 2 and 3 judged A2 to be more preferable than A1 on two of the three criteria and the magnitudes of their judgments, when synthesized, results in A2 being more preferred to A1 (despite the magnitude of the one judgment where A1 was judged to be more preferable than A2).

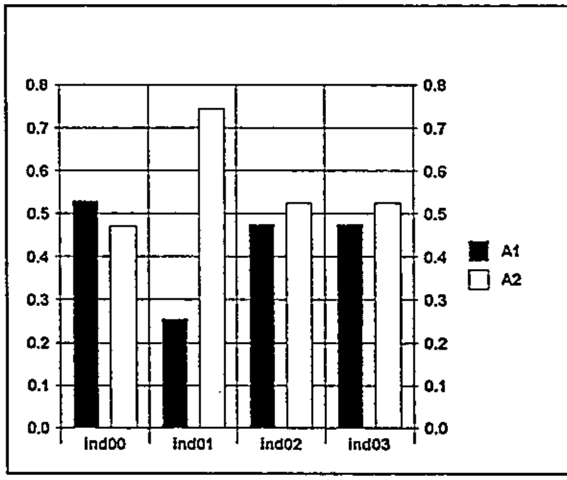

Figure 8 -- Aggregating judgments (AIJ)

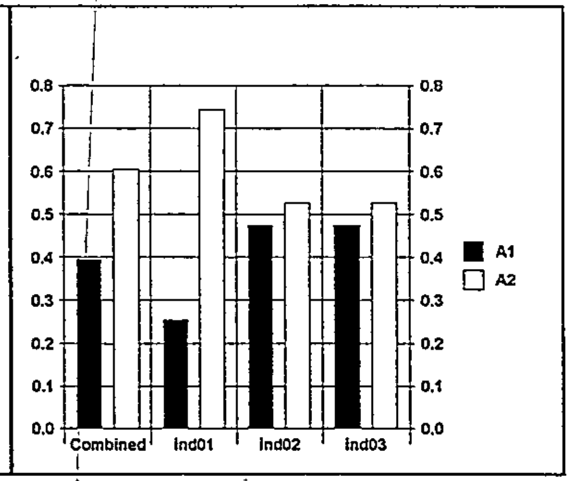

Figure 9 -- Aggregating of priorities (AIP)

The aggregation of the judgments is shown in Figure 8 (the '00' result is the aggregation of individual judgments or AIJ) while the aggregation of individual priorities (AIP) is shown in the 'combined' section of Figure 9. The results of the aggregation of judgments contradicts the Pareto principle since A1 is preferred to A2 while, individually, each of the individuals prefer A1 to A2! However, violating the Pareto principle cannot occur when we aggregate individual priorities, as is illustrated in the Figure 9 results. The reader can verify that no violation of the Pareto principle will occur with Ramanathan and Ganesh's example either, when the AIP method is used.

If this problem is in context where individuals are willing or must relinquish their own preferences (values, objectives) for the good of the organization then synthesizing individual priorites for the alternatives is irrelevant and the Pareto principle does not apply. If this is problem is in context where individuals are each acting in his or her own right, with different value systems, AIP should be used and the Pareto principle will not be violated. The context, rather than mechanics is paramount.

Weighted Geometric Means

When calculating the geometric average of the priorities (AIP) or judgments (AIJ) we take the nth root of the product or priorities or judgments. If, however, group members are not equally important, we can form a weighted geometric average by taking the product of the priorities (AIP) or judgments (AIJ), each raised to the power of the priority of the decision maker. For example, in the AIP case,

$$P_g(A_j)=\sum_{i=1}^{n} w_i P_i(A_j)$$

where:

$P_g(A_j)$: Refers to the group priority of alternative j

$P_i(A_j)$: Refers to individual i's priority of alternative j

w_i: is the weight of individual i, and

n: is the number of decision makers.

The question arises as to how to compute the w_i's. Saaty (1994, p219) suggests forming a hierarchy of factors such as expertise, experience, previous performance, persuasive abilities, effort on the problem, etc. to determine the priorities of the decision makers. But who is to provide judgments for this hierarchy? If it can not be agreed that one person (a supra decisionmaker) will provide the judgments, it is possible to ask the same decision makers who provided judgments for the original hierarchy to provide judgments for this hierarchy as well. If so, we have a meta-problem of how to weight their individual judgments or priorities in the aggregation process to determine the weights for the decision makers to apply to the aggregation of the original hierarchy. One possibility is to assume equal weights. Ramanathan and Ganesh provide another method which they call the eigenvector method of weight derivation. They reason that, if $\overline{w} = (w_1, w_2, \ldots, w_n)^t$ is the 'true' (but unknown) weight priority vector for the individual's weights, and if the individual weight priority vectors derived from the judgments from each of the individuals are arranged in a matrix: $\overline{M} = (\overline{m}_1, \overline{m}_2, \ldots \overline{m}_n)$, then we can aggregate to find the priorities of the individuals, $\overline{x}$, where:

$\overline{x} = \overline{M} * \overline{w}$. Then Ramanathan and Ganesh reason that $\overline{x} = \overline{w}$, resulting in the eigenvector equation: $\overline{w} = \overline{M} * \overline{w}$.

We observe that this method is reasonable *only if* the weights for obtaining priorities of the decision makers are assumed to be the same as the weights to be used to aggregate the decision makers judgments/priorities for obtaining the alternative priorities in the original hierarchy. In general, this need not be the case.

Summary and Conclusions

When several individuals provide judgments with the Analytic Hierarchy Process, one may aggregate individual judgments (AIJ) or aggregate individual priorities(AIP).

When group members are not acting as individual "experts", but instead act in concert, pooling their judgments in order to develop more accurate comparisons, one should take the geometric average of their individual judgments (AIJ). The geometric averaging of individual judgments satisfies the reciprocity requirement, implying a synergistic aggregation of individual preferences in such a way that the group becomes a new 'individual' and behaves like one. Individual identities are lost with every stage of aggregation and the Pareto principle is irrelevant. When group members are each expert in his or her own right and act as individuals, one should take the geometric average of their resulting priorities (AIP). The Pareto principle will not be violated.

If the group members are not considered to be of equal importance, a weighted geometric average can be employed with either AIJ or AIP. A separate hierarchy can be constructed to derive priorities for the decision makers. There is great flexibility in determining who makes the judgments for this hierarchy. In the case where the original group members are the ones to make these judgments, Ramanathan and Ganesh's eigenvector method can be used *provided* the priorities of the decision makers in aggregating to obtain decision maker priorities are assumed to be the same as the priority of the decision makers in aggregating the original problem hierarchy.

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