The Combination of AHP and DEA, and its Application in Evaluating Economical and Social Systems

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Abstract

AHP (Analytic Hierarchy Process) and DEA (Data Envelopment Analysis) are multiple objective decision making methods. By dividing the decision factors into levels and using judgment intensities in pairwise comparisons, AHP can set the relative priorities of alternatives, whereas, the linear programming method uses the concept of forward positive plan of production, and DEA can obtain the relative efficiency of decision making units. Since the 1970's, both methods have developed quickly and have been applied widely in economical analysis. In this paper, we will analyze the characteristics of AHP and DEA and the relations between them. We will then combine AHP with DEA to obtain a more practical method of evaluation. Finally, we will provide the results of evaluating 28 cotton textile mills and 52 rubber and plastic mills of china. In addition to the development of the national economy, people are interested in the problems of relative efficiency of enterprises or departments, fund allocation, etc. To meet the practical needs of decision making, in the early 1970's, professor T.L. Saaty proposed AHP. Due to its simplicity, ease, practicability and reasonableness, AHP has been widely applied in practice. Simultaneously, the American operational researchers A. Charnes and W.W. Cooper proposed the mathematic model, Data Envelopment Analysis (DEA); DEA estimates the relative efficiency of enterprises or departments. Currently, both methods are being further developed and applied by departments. It is an efficient way to combine the methods would provide an efficient way to obtain decision-making methods in different situations. By analyzing the characteristics of DEA and AHP, we will explore a way to combine them to achieve a more practical evaluating method.

1. Introduction of DEA

Consider n enterprises or decision making units (DMU) each with m inputs and s outputs and we would like to evaluate them according to their relative efficiency. Let $x_j = (x_{1j}, \ldots, x_{mj})^T$ and $y_j = (y_{1j}, \ldots, y_{sj})^T$, $j=1,2,\ldots,n$ be the input and output vectors, respectively, of the jth unit and let the Efficiency Index (E.I.) be given by:

$$h_j = (\sum_{r=1}^{s} u_r y_{rj}) / (\sum_{i=1}^{m} v_i x_{ij}) \quad j=1,2,\ldots,n \tag{1}$$

where $v = (v_1, \ldots, v_m)^T \geq 0$ and $u = (u_1, \ldots, u_m)^T \geq 0$, are the weight vectors of inputs and outputs, respectively. The efficiency index h_j can be less than or equal to 1 by appropriately selecting u and v.

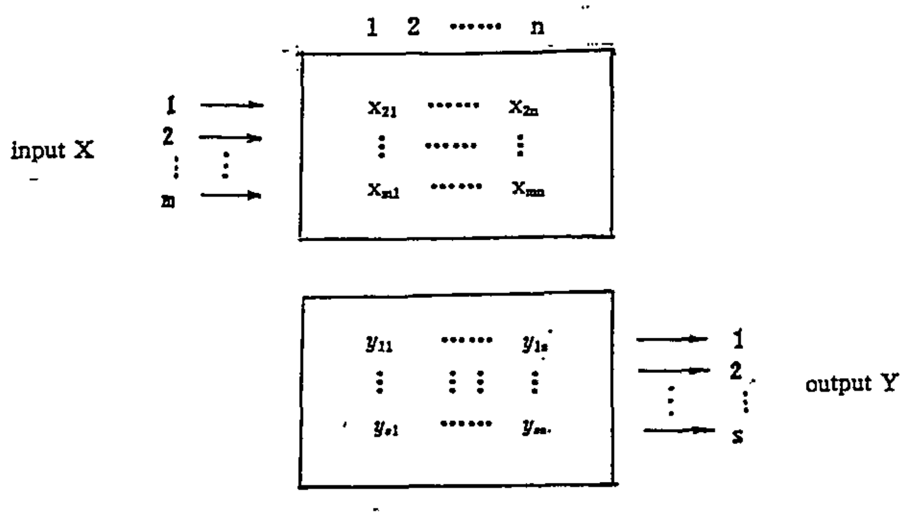

Figure 1

A. Charnes, W.W. Cooper and E. Rhodes proposed the following fractional programming model to evaluate the j_0th unit:

$$\begin{aligned} & \max\ (u^T y_0)/(v^T x_0) \\ (p)\quad & \text{s.t.},\ (u^T y_j)/(v^T x_j) \le 1,\ j=1,2,\ldots,n \\ & u\ge 0,\ v\ge 0 \end{aligned} \tag{2}$$

where x_0 and y_0 are the input and output vectors, respectively, of the j_0th unit,. Let u and v be the optimal solution of (2). Let $t=1/v^T x_0$, $\omega = vt$, $\mu = ut$. (2) is now transformed into a linear programming problem given by:

$$\begin{aligned} & \max\ \mu^T y_0 = v_p \\ (p)\quad & \text{s.t.},\ \omega^T x_j - \mu^T y_j \ge 0,\ j=1,2,\ldots,n \\ & \omega\ge 0,\ \mu\ge 0 \end{aligned} \tag{3}$$

where ω and μ have the same mean as v and u. This model is called the CCR model.

Definition 1. If the solution(ω,μ) of (3) satisfy $v_p = \mu^{0T} y_0 = 1$, then the DMU is called a <u>weak DEA efficient unit</u> (CCR). In addition, if the solution of (3) is positive, then the DMU j_0 is called a <u>DEA efficient unit</u>.

Let $x = (x_1, \ldots, x_m)^T$, $y = (y_1, \ldots, y_s)^T$, $f_i(x,y) = x_i$, $i=1,2,\ldots,m$, and $f_{m+j}(x,y) = -y_j$, $j=1,2,\ldots,s$. The multiobjective programming problem is given by:

$$V - \min(f_1(x,y), \ldots, f_{m+s}(x,y))$$

$$(V_p) \quad s.t., \qquad (4)$$

$$(x,y) \in \hat{T} = \{(x_1,y_1), \ldots, (x_n,y_n)\}.$$

If we extend $\hat{T}$ to the set T given by:

$$T = \{(x,y) \mid \sum_{j=1}^{n} x_j\lambda_j \le x, \sum_{j=1}^{n} y_j\lambda_j \ge y, \lambda_j \ge 0, j=1,2,\ldots,n\}$$

we obtain the following multiobjective programming problem:

$$V - \min(f_1(x,y), \ldots, f_{m+s}(x,y))$$

$$(V_p) \quad s.t., \qquad (5)$$

$$(x,y) \in T.$$

Thus we can prove the following:

Theorem 1. If the decision making unit j_0 is an (weak) efficient unit, then (x_0,y_0) corresponding to j_0 is a (weak) pareto efficient solution of problem (V_p), and vice versa.

Proof. (see [2]).

We can solve the DEA problem instead of solving multiple objectives programming problems.; and from the DEA efficiency we can obtain the frontier plans. Here the frontier plans of production are the plans which consist of pareto efficient solutions. These plans represent the frontier levels of production at present. From the position of DMU relative to the frontier plan, we can get information on improving the DMU (for example, which input should be decreased and which output should be increased). This is an important advantage of the DEA model.

However, the above model has two weaknesses. First, all inputs (outputs) are considered equally important. For example, if there are two products, coal and gold, then they will have the same value in this model. Perhaps increasing the output of coal to 100 tons may be more efficient than increasing the output of gold to one ton. This is not in agreement with empirical results. Thus the inputs and outputs should be prioritized. This means that the feasible range will be reduced. Second, if the number of inputs and outputs is large, solving the linear programming problem (p) will be time consuming, and it is likely that the number of DEA efficient units will increase. This will decrease the value of the evaluation. We should consider these problems further to improve the model (CCR).

2. Characteristics of AHP in Evaluation

When we apply the AHP to analyze the benefit/cost of a decision, we usually consider two processes: ranking the benefits, and ranking the costs. We start by considering the problem of ranking the benefits. First, we use the judgment intensities in pairwise comparisons to set the weight vector $u = (u_1, \ldots, u_s)^T$ for s goals, then for each goal we can get the weight vector $(y_{i1}, \ldots, y_{in})$. We write:

$$Y = \begin{bmatrix} y_{11} & \cdots\cdots & y_{1n} \\ \vdots & \vdots & \vdots \\ y_{s1} & \cdots\cdots & y_{sn} \end{bmatrix} = (y_1, \cdots, y_n)$$

where y_j is the weight vector of the jth unit with respect to the goals. Let $y_{i0} = y_0$. The benefit index of the j_0th unit is given by:

$$(w_0)_y = \sum_{r=1}^{s} u_r y_{r0} = u^T y_0$$

Similarly, the cost index of the j_0th unit is given by:

$$(w_0)_s = \sum_{i=1}^{m} v_i x_{i0} = v^T x_0$$

Thus, the relative Efficiency Index is given by:

$$w_0 = u^T y_0 / v^T x_0$$

In some cases, x_0 and y_0 can be obtained directly from normalizing the input and output data. Thus, we can conclude that the results of AHP and DEA have the same meaning. The only difference between AHP and DEA is how the weight vectors u and v are determined. The vectors u and v obtained by AHP depend on the judgments of decision makers which are the same for all units. With AHP, we cannot obtain the relative efficiency required to attain present production levels, but the E.I.'s order only. This implies that there is only one optimal DMU, but the relationship between the individual and the total level is unclear.

In AHP, we have considered the partiality of different goals and obtain the weight. In fact, because of the complication of economic factors, the multiplicity and fluctuation of objectives, and the opinions of experts it is unsuitable to fix the same weight vectors u and v for all units. Moreover, some important intensities can not be described very precisely. The same goal is not interpreted the same by different DMU's but also within the same DMU. This suggests that we should assign weights to have a changeable range and that it is more reasonable to consider the relative efficiency in this range.

3. Using the weighting matrix of AHP in CCWH model (A-CCWH)

To improve the CCR model Q.L. Wei [2] proposed the "Cone-Ratio" DEA model named CCWH model given by:

$$(P_w) \quad \begin{aligned} &\max \ \mu^T y_0 \\ &\text{s.t.,} \ \omega^T x_j - \mu^T y_j \geq 0, \ j=1,2,\ldots,n \\ &\omega^T x_0 = 1 \\ &\omega \in V, \ \mu \in U \end{aligned} \qquad (6)$$

where U and V are convex polyhedral cones. If the convex cone V is formed by $a^{(1)}, \ldots, a^{(m')}$ then any nonnegative linear combination of the $a^{(i)}$, $i=1,2,\ldots,m'$, is also an element of V,

$$V = \{\sum_{i=1}^{m'} a^{(i)} w_i | w_i \geq 0, \ i=1,2,\ldots,m'\} \qquad (7)$$

Let $A = (a^{(1)T}, \ldots, a^{(m')T})^T$ and $B = (b^{(1)T}, \ldots, b^{(s')T})^T$ be two mappings from $\mathbb{R}_m^+$ and $\mathbb{R}_s^+$ into V and U, respectively. We have

$$V = \{A^T \omega' | \omega' \geq 0\}$$
$$U = \{B^T \mu' | \mu \geq 0\} \qquad (8)$$

For every $\omega \in V$ and $\mu \in U$ there are $\omega' \in \mathbb{R}_m^+$ and $\mu' \in \mathbb{R}_s^+$ such that:

$$\omega = A^T \omega' \ \text{and} \ \mu = B^T \mu'. \qquad (9)$$

Substituting (9) into equation (6) we have:

$$\begin{aligned} &\max\ \mu^{T}(By_0) \\ (P_w)\quad &\text{s.t.},\ \omega'^{T}(Ax_j) - \mu'^{T}(By_j) \geq 0,\ j=1,2,\ldots,n \qquad (6) \\ &\omega'^{T}(Ax_0) = 1,\ \omega'\geq 0,\ \mu'\geq 0 \end{aligned}$$

Thus, after weighting the primary data by the weight matrices we get the result of the CCWH model.

Moreover, the number of variables will change from m+s to m'+s'. If m'+s'<m+s, then the scale of the problem will be reduced. It can be proven that when the decision making unit j_0 is DEA (CCWH) efficient, it must be DEA (CCR) efficient, the converse may not be true. Thus the number of DEA (CCWH) efficient units may be less that the number of DEA (CCR) efficient units. The weighting matrices A and B can be obtained from AHP. We will refer to the mixed model as A-CCWH.

4. Applications

(A) Evaluating the economic benefits for 38 cotton textile mills.

The output goals are:

(1) Income from selling (RMB million)
(2) Total sum of profit and tax (RMB million)
(3) Total industrial output value (RMB million)
(4) Industrial net product (RMB million)
(5) Total labor productivity (RMB 1000/head)

The input goals are:

(1) Yearly average balance of quota current funds (RMB million)
(2) Yearly average balance of net value of fixed assets (RMB million)
(3) Yearly average number of employees (100 head)
(4) Energy consumption of industrial production (1000 T stan. coal)
(5) Plant cost of products sales (RMB million)
(6) Original value of fixed assets (RMB million)

First we applied the CCR model directly to evaluate and obtain 15 efficient DMU's in 38 mills. The number of efficient DMU's is too large. We then applied A-CCWH with the weighting matrices (as follows) A_{1B} is given to the first expert to evaluate. The number of efficient DMU is reduced to one (16th mill). This result is in accord with the result of AHP. Because the matrices are quite consistent, the vertex angles of U,V are small. This causes the changeable ranges of U,V to be small and the efficient units to be sharply reduced.

$$A_1^T = C_1^{(1)} = \begin{bmatrix} 1 & 2 & 2 & 1/4 & 1/5 & 3 \\ 1/2 & 1 & 2 & 1/5 & 1/7 & 2 \\ 1/2 & 1/2 & 1 & 1/5 & 1/5 & 3 \\ 4 & 5 & 5 & 1 & 1/5 & 5 \\ 5 & 7 & 5 & 5 & 1 & 7 \\ 1/3 & 1/2 & 1/3 & 1/5 & 1/7 & 1 \end{bmatrix}$$

$$D \; J \quad B_1^T = C_I^{(1)} = \begin{bmatrix} 1 & 1/7 & 1/3 & 1/4 & 1/5 \\ 7 & 1 & 7 & 6 & 1 \\ 3 & 1/7 & 1 & 1 & 1/5 \\ 4 & 1/6 & 1 & 1 & 1/5 \\ 5 & 1 & 5 & 5 & 1 \end{bmatrix}$$

The Efficiency Index of the 26th mill which is efficient in CCR is reduced to 0.6621. Because the judgments are essentially in agreement, the result obtained from the group is almost the same as the result from a single expert. The input-output data and results of the mills 16, 26, 13 and 18 are listed in Tables 1 and 2. From Tables 1 and 2, it can be clearly seen that mill 16 has higher outputs and lower inputs than mills 13 and 18. So the 16th mill is always efficient. Even though mill 26 has higher inputs than the 16th, it is still efficient in the CCR model, owing to its 4 higher outputs. However, the weight of its dominant output is lowest in the A-CCWH model, this fact makes its E.I. reduce sharply in A-CCWH.

Table 1

mill	input						output				
	1	2	3	4	5	6	1	2	3	4	5
13	18.3	34.4	51.0	10.2	71.6	49.1	93.8	19.8	58.5	29.5	11.8
16	12.1	17.2	50.0	5.0	77.3	37.4	107.7	26.5	131.9	41.8	26.4
18	25.1	41.1	59.8	22.8	136.3	70.6	160.4	24.8	115.2	41.0	20.1
26	21.4	30.4	95.4	36.4	145.1	71.1	191.1	40.7	157.2	55.8	18.2

Table 2

mill	efficient index		
	CCR	A—CCWH	AHP m
13	0.9121	0.6465	0.4255
16	1.0000	1.0000	1.0000
18	0.9037	0.5610	0.4195 m
26	1.0000	0.6621	0.4717

(B) Evaluating economic benefits for rubber and plastic mills

This is an evaluation for transtrade mills. There are 17 targets related, 9 targets for input and 8 targets for output. The number of mills related to evaluate is 52. When we applied CCR model directly, we get 21 efficient mills, about 40% in total. Then we applied the A-CCWH model with the first experts judgement matrices; the efficient mill are reduced to four. The E.I. of some of the efficient mills in CCR is reduced to 0.2104 in A -- CCWH. The results of the group judgement obtained from four experts are roughly the same.

In sum, the model A-CCWH combining AHP and DEA is a more practical evaluating and optimum seeking method. We expect it will play a role in evaluation to help raise the economic benefits of enterprises.

References

(1) T.L. Saaty: "The Analytic Hierarchy Process," Pittsburgh: *RWS Publications*, 1988.

(2) Q.L. Wei: "The DEA Model for Evaluating Relative Efficiency--A New Operations Research Branch," *Publishing House of People's University of China*. 1987.