ESTABLISHING THE HIERARCHY STRUCTURE OF AHP AND ITS IMPLEMENTING ON COMPUTER

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ABSTRACT

when system decision is made by AHP, the first question to be faced is how to analyze a complicated system and construct a reasonable hierarchy structure. Usually such a hierarchy structure merely depends on analysis and judgment of decision makers; however, using this method not only takes much time but easily makes confusion among hierarchies also. A definition about hierarchy structure is proposed at first by the author, which includes various structure forms. And then subordinate relationships of system elements are described by a directed graph. With mathematic methods, it can be conveniently changed into a reachability matrix. Finaly, according to the definition, hierarchies are partitioned and a hierarchy structure is obtained.

The process mentioned above can be made either by listing tables or by implementing on computer. A practical example and the steps of applying with computers are also given in this paper.

I . Formal Description

According to the normal definition, a sub-hierarchy is not thought to be independent. For convenience to comput, in this paper, it is considered as a independent hierarchy.

Definition 1. Dual relationship '$\leqslant$' is said to be a dependency, and relationship '$x<y$' means that $x<y$ and $x\neq y$. y is said to cover (dominate) x, if $x<y$ and $x<t<y$ is possible for no t.

H is a nonempty set. For every element $x \in H$, the set x^- that implies all elements dominated by x is called the set dominated by x , and the set x^+ that implies elements dominating over x is called the set dominating over x, i. e.

$$x^- = \{ y \mid x \text{ covers } y, \forall y \in H\} \qquad (1)$$
$$x^+ = \{ y \mid y \text{ covers } x, \forall y \in H\}$$

Definition 2. Let H is a finite partially ordered set with largest element b. H is a hierarchy structure if it satisfies the conditions:

(1). H is partitioned into sets L_k, $k = 1, 2, \ldots m$ for some m, where $L_1 = \{b\}$;

(2). For any $x \in L_k$ $(1\leqslant k\leqslant m-1)$, x^- is nonempty, and either for every $x \in L_k$, $x^- \subseteq L_{k+1}$ exists , or for certain $x \in L_k$ exists $i>1$ that makes $x^- \cap L_{k+j} = \varnothing$ for every $j<i$ but $x^- \subseteq L_{k+i}$ $(k+i\leqslant m-1)$. In the latter case the levels L_{k+j}, $j=1, 2, \ldots i-1$ is called sub-hierarchy also;

(3). For any $x \in L_k$ $(2\leqslant k\leqslant m)$, x^+ is nonempty, and either $x^+ \subseteq L_{k-1}$ or for certain $y \in x^+$ exists $i>1$ that makes $y \in L_{k-i}$ $(k-i\geqslant 1)$. In the latter case , L_{k-j}, $j=1, 2, \ldots i-1$ all are sub-hierarchies. Except the sub-hierarchies, all of rest hierarchies defined are called principal hierarchy.

According to the definition, it is not difficult to obtain following properties:

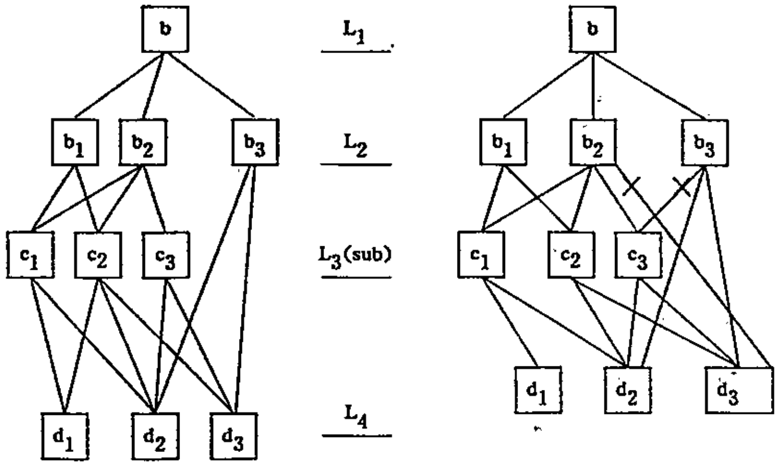
(1). Every element of H belongs and only belongs to one level and the intersection of two levels is empty;

(2). There is no dominated relation between two elements in the same level, i. e. all elements in interior of a level are independent;

(3). Every element in L_k ($1<k\leqslant m$) must be at least dominated by an element in a level over L_k; every element in L_{k-1} ($2\leqslant k<m$) is only able to dominate over some elements in one level below L_{k-1}, but it is impossible that an element can dominate over elements in two or more than two deferent levels;

(4). Except the sub-hierarchies, there is no dominating relation between two elements in two inconsecutive levels.

Figure 1 (a) has given a hierarchy structure, which includes a sub-hierarchy. Figuer 1 (b) is not a hierarchy structure, because the element b_2 (and b_3) has dominated over the elements c_3, d_3, in deferent levels respectively. It is clear to see that $d_3\in b_2^-$ and $d_3\in L_4$ but $b_2^-\cap L_3=\{c_1, c_2, c_3\}=\varnothing$. Removing the lines printed '$\times$' in Fig 1 (b), it will become a hierarchy structure.



(a) a hierarchy structrue (b) a non—hierarchy structrue

Fig 1

In a complex system, however the main dependency among elements can be easy to obtain from analysis and judgment, but because of the complication of condition the dominative relations are not easy to be known, and it is just the key to determine the hierarchy structure. In the following, we will apply the directed graph and reachability matrix to make an approch to this problem.

II. The Directed Graph and Calculating of Reachability Matrix

Let $G = \langle V, E \rangle$ represent a directed graph, here $V=\{v_1, v_2, \ldots v_n\}$ is the nodal set and $E=\{l_1, l_2, \ldots l_m\}$ is the frontier set. Every edge can be represented by a pair of nodes, for example, $l_i=(v_j, v_k)$, and the direction of l_i is from v_j to v_k.

Assume the nodes is arranged in a given order, for example, in the ordinal number of nodes. This assumption will be followed in this paper. Thus the directed graph will be related to an adjacency matrix A

$$A = (a_{ij}) \tag{2}$$

$$a_{ij}=\begin{cases}1 & \text{if } (v_i, v_j)\in E\\ 0 & \text{if } (v_i, v_j)\,\bar{\in}\, E\end{cases}$$

The adjacency matrix A gives all entries from one node to another, whose length all is unit. In a (n, m) directed graph every essential entry can go across n nodes at most, so its length is less than or equal to n-1 units and the length of essentail circuit is less than or equal to n units. It is clear that if $a_{ik} a_{kj} \neq 0$ for adjacency matrix, then there is an entry from v_i to v_j which go across v_k and its length equal to two units. So the value of $a_{ik} a_{kj}$ is the number of entries being two units in length, i. e. the element a_{ij} of A^2 gives the number of entries from v_i to v_j being two units in length. And like this, the elements of $A^3, \ldots A^n$ give the number of entries respectively being 3, . . . n units in length. Thus let matrix $R = (r_{ij})$, here

$$R = A + A^2 + \ldots + A^n \tag{3}$$

The element r_{ij} of matrix R is just equal to the sum of entries and circuits from v_i to v_j, whose length is less than or equal to n. If there is an entry or circuit from v_i to v_j in directed graph, then it is said that is accessible from v_i to v_j. Thus

when $r_{ij} = 0$, it is inaccessible from v_i to v_j;
when $r_{ij} \neq 0$, it is accessible from v_i to v_j;
when $r_{ii} = 0$, there is no cricuit across v_i.

Because we are only interested in reachability between nodes instead of the length of entries, we can construct a matrix $P = (p_{ij})$, here

$$p_{ij} = \begin{cases} 1, & \text{if } r_{ij} \neq 0 \\ 0, & \text{if } r_{ij} = 0 \end{cases} \qquad i, j = 1, 2, \ldots n \tag{4}$$

The matrix P is called the reachability matrix of graph $G = \langle V, E \rangle$. If $p_{ij} = 1$, then it is accessible from v_i to v_j; otherwise it is inaccessible. The reachability matrix P can be obtained by Boolean operation of adjacency matrix A, i. e.

$$P = A \oplus A^{(2)} \oplus \ldots \oplus A^{(n)} \tag{5}$$

here $A^{(i)}$ is the ith power of A in the sense of Boolean, i. e. $A^{(i)} = A^{(i-1)} \otimes A$ and it is stipulated that

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 1, \quad 1 \oplus 1 = 1$$
$$0 \otimes 0 = 0, \quad 0 \otimes 1 = 0, \quad 1 \otimes 0 = 0, \quad 1 \otimes 1 = 1$$

In the course of operation of P, if it exsits certain $k < n$ that makes $A^{(k+1)} = A^{(k)}$, then $A^{(j)} = A^{(k)}$ for any $j > k$. Thus

$$P = A \oplus A^{(2)} \oplus \ldots \oplus A^{(k)} \tag{6}$$

The calculation of P can be done by recurrence formula. let $P_1 = A$, $M_{k+1} = P_k \otimes A$. If $M_{k+1} = P_k$, Then $P = P_k$, otherwise $P_{k+1} = P_k \oplus M_{k+1}$. And Calculating P_{k+1}, it should be only considered to put the new non-zero elements which occur in M_{k+1} into P_k. In other words, Let $P_k = (p_{ij}^{(k)})$, construct matrix $Q_{k+1} = (q_{ij}^{(k+1)})$, here

$$q_{ij}^{(k+1)} = \begin{cases} 1, & \text{when both } m_{ij}^{(k+1)} = 1 \text{ and } p_{ij}^{(k)} = 0 \text{ exist;} \\ 0, & \text{otherwise;} \end{cases} \tag{7}$$

then

$$\begin{aligned} &P_{k+1} = P_k \oplus Q_{k+1} \\ &P = P_k \quad \text{when } Q_{k+1} = 0. \end{aligned} \tag{8}$$

The following example can be explained the process of calculation.
Example 1. Find the reachability matrix of the graph in Fig 2.
Solving: The adjacency matrix of Fig 2 is A

$$A=\begin{bmatrix}0&1&0&0&0\\1&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix}$$

$$P_1=A. \qquad M_2=P_1\otimes A=\begin{bmatrix}1&0&1&0&0\\0&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} \qquad Q_2=\begin{bmatrix}1&0&1&0&0\\0&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$P_2=P_1+Q_2=\begin{bmatrix}1&1&1&0&0\\1&1&1&0&0\\1&1&1&0&0\\0&0&0&1&1\\0&0&0&1&1\end{bmatrix} \qquad M_3=P_2\otimes A=\begin{bmatrix}1&1&1&0&0\\1&1&1&0&0\\1&1&1&0&0\\0&0&0&1&1\\0&0&0&1&1\end{bmatrix}$$

$$Q_3=0 \qquad P=P_2$$

In this process, calculating M_k has to do Boolean operation of matrix yet, so it is still complex. We will simplify the process as following:

(1). Let $(P_1)_i=\{\, j \mid a_{ij}=1,\ 1\leqslant j\leqslant n\,\}$, $i=1, 2, \ldots n$, $k=1$

(2). Let

$$(Q_{k+1})_i=\begin{cases}\{\, l \mid l\in(P_1)_j,\ j\in(P_k)_i \text{ and } l\ \bar{\in}\ (P_k)_i\} & \text{when } (P_k)_i\neq\emptyset;\\ 0, & \text{when } (P_k)_i=\emptyset\end{cases}$$

(3). Give a judgment on $(Q_{k+1})_i$, $i=1, 2, \ldots n$. If they are all enpty, then $P=(p_{ij})$ has been obtained, where

$$p_{ij}=\begin{cases}1, & \text{if } j\in(P_k)_i\\ 0, & \text{if } j\ \bar{\in}\ (P_k)_i\end{cases}$$

else let

$(P_{k+1})_i=(P_k)_i\cup(Q_{k+1})_i$, $k=k+1$, go to 2.

This process can be performed by tabulating as well as by computer. Table 1 has listed the process calculating reachability matrix of adjacency matrix A of example 1.

Table 1

number of node i	(P_1)	(Q_2)	(P_2)	(Q_3)
1	2	1, 3	1, 2, 3	0
2	1, 2	2	1, 2, 3	0
3	2	1, 3	1, 2, 3	0
4	5	4	4, 5	0
5	4	5	4, 5	0

then

$$P=\begin{bmatrix}1&1&1&0&0\\1&1&1&0&0\\1&1&1&0&0\\0&0&0&1&1\\0&0&0&1&1\end{bmatrix},$$

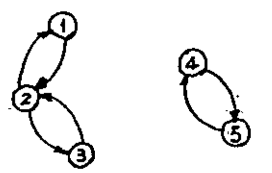


Fig 2

Ⅲ. Establishing Hierarchy Structure of AHP

For a given system, we can represent the subordinate between the elements by directed graph. Every node in directed graph represents an element and the directed edge (v_i, v_j) represents v_j dependent on v_i. From the directed graph and its adjacency matrix, the reachability matrix P can be got by the process mentioned above. Following conditions must be met for a reachability matrix of a hierarchy structure:

(1). $p_{ii}=0$, for $i=1, 2, \ldots n$. It shows that no any circuits of v_i exist. this means that there is no dependent relation for every node itself.

(2). If $p_{ij}=1$, then $p_{ji}=0$, for $i, j=1, 2, \ldots n$. Otherwise, if $p_{ji}=1$, then $p_{ij}=0$. This means that only one directed edge between v_i and v_j can be adjoined, so there is no dependence or only one dependence betwwen two elements.

(3). The reachability matrix must be irreducible, otherwise, the system may be decomposed in several independent systems.

If the conditions 1, 2 above are not be met, then the system is a feedback system or a hierarchy structure with inter-dependence. It is out of the range discussed by this paper. So if we will treat system as a ordinary hierarchy structure, we must exam and recorrect the directed graph so that its reachability matrix meet the conditions above.

Definition 3. Let $G=\langle H, E\rangle$ is a directed graph of a system, where $H=\{v_1, v_2, \ldots v_n\}$, elements v_i is called entecedent of v_j, if v_j can be reached from v_i, and v_j is called consequent of v_i.

Let $R(v_i)$ is the ordinal number set of consequent of v_i, $A(v_i)$ is the ordinal number set of entecedent of v_i, i. e.

$$R(v_i)=\{j \mid v_j\in H, p_{ij}=1\}$$
$$A(v_i)=\{j \mid v_j\in H\ p_{ji}=1\} \tag{9}$$

From the definition of hierarchy structure, there is only one element in the highest level. Let it is v_J, then $A(v_J)=\varnothing$ thus $P_{iJ}=0, i=1, 2, \ldots n$.. It means all of the elements in Jth column are equal to zero. Inversely, if the elements all of Jth column are zero, then $A(v_J)=\varnothing$, and $v_J\in L_1$. From this we can conclud that 'there is one and only one column which elements all are zero in a reachability matrix.'

Let $H_1=H$, $P^{(1)}=P$, crossing off the Jth column whose elements all are zero and the Jth row, preserving the ordinary order of rest columns and rows, the matrix $P^{(1)}$ will be deflated to $P^{(2)}$. It correspond to deleting L_1 from H. In the set consisted by rest elements of H_1, i. e. $H_2=H_1-L_1$, the elements (may not only one) whose entecedent set is empty must be belonged to L_2, i. e. the nodes corresponding the columns whose elements all are zero are in L_2 and so on. In general, if $P^{(k+1)}$ is the matrix having deflated for k times, then the entecedent of its elements '.' is writen by $A_{k+1}(.)$. And for $\forall\ v_i\in H_{k+1}=H-\sum_{i=1}^{k}L_i$, if $A_{k+1}(v_i)=\varnothing$, then $v_i\in L_{k+1}$; otherwise $v_i\bar{\in} L_{k+1}$. By this method the hierarchy structure can be obtained. Beside this, we need draw the connecting lines between the elements in deferent levels yet. It is able to do if the dominative relation between elements are known. In fact, for every $v_i\in L_k$, the set of elements dominated by v_i, i. e. v_i^-, is

$$v_i^-=\begin{cases} R(v_i)\bigcap L_{k+1}, & \text{when } R(v_i)\bigcap L_{k+1}\neq\varnothing \\ R(v_i)\bigcap L_{k+j}, & \text{when } R(v_i)\bigcap L_{k+l}=\varnothing, l=1, 2, \ldots j-1 \\ & \text{but } R(v_i)\bigcap L_{k+j}\neq\varnothing \end{cases} \tag{10}$$

Connecting the elements with dominative relation by lines the graph of hierarchy structure can be obtained. The steps from reachability matrix to obtain the hierarchy and the dominative relation are given in following:

(1). Let $k=1$, $P^{(1)}=P$, $R(v_i)=(P)_i$, $i=1, 2, \ldots n$;

(2). Find the columns whose elements all are zero in $P^{(k)}$. If the ordinal number of these column are $k_1, k_2, \ldots k_s$, then $L_k=\{v_{k1}, v_{k2}, \ldots v_{ks}\}$.

(3). Crossing off the columns and rows of k_1 th, $\ldots k_s$ th from $P^{(k)}$, $P^{(k+1)}$ is obtained;

(4). If $P^{(k+1)}=0$, then the hierarchies $L_1, L_2, \ldots L_k$ have been obtained. Then go to (5); or else let $k=k+1$ go to (2);

(5). Following steps are to find the dominative set of every element. Let $m=1$:

(6). For every $v_i\in L_m$, let $l=1$:

(7). Find $v_i^-=R(v_i)\bigcap L_{m+l}$;

(8). If $v_i^-=\varnothing$, let $l=l+1$, go to (7). if $v_i^-\neq\varnothing$, then go to (6) and go on with to find the dominative set of rest elements of L_m. If their all are obtained, then go to (9) ;

(9). Let m=m+1. if m=k, stop; or else go to (6) to find the dominative set of elements in next level.

Finally, we will give a example of computation here.

Example 2. A directed graph of system is given in Fig 3. Let us draw its hierarchy structure.

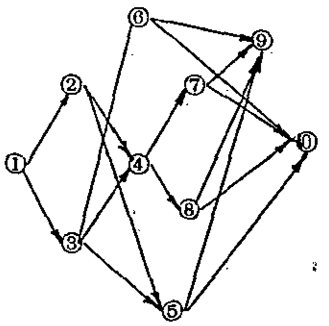


Fig 3

Sovlving:

(1). Calculating the reachability matrix with listing table.

Table 2

i	$(P_1)_i$	$(Q_2)_i$	$(P_2)_i$	$(Q_3)_i$	$(P_3)_i$	$(Q_4)_i$
1	2, 3	4, 5, 6	2, 3, 4, 5, 6	7, 8, 9, 10	2, 3, 4, 5, 6, 7, 8, 9, 10	∅
2	4, 5	7, 8, 9, 10	4, 5, 7, 8, 9, 10	∅	4, 5, 7, 8, 9, 10	∅
3	4, 5, 6	7, 8, 9, 10	4, 5, 7, 8, 9, 10	∅	4, 5, 6, 7, 8, 9, 10	∅
4	7, 8	9, 10	7, 8, 9, 10	∅	7, 8, 9, 10	∅
5	9, 10	∅	9, 10	∅	9, 10	∅
6	9, 10	∅	9, 10	∅	9, 10	∅
7	9, 10	∅	9, 10	∅	9, 10	∅
8	9, 10	∅	9, 10	∅	9, 10	∅
9	∅	∅	∅	∅	∅	∅
10	∅	∅	∅	∅	∅	∅

(2). The reachability matrix $P=P_3$ is feasible.

$$P = P_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3). Partition elements into some levels and find the dominative set of elements with listing table 3.

Table 3

k	the elements of L_k	$R(v_i) = (P_j)_i$	$v^- = R(v_i) \cap L_{k+j}$	j
1	1	2, 3, 4, 5, 6, 7, 8, 9, 10	2, 3	1
2	2 3	4, 5 7, 8, 9, 10 4, 5, 6, 7, 8, 9, 10	4, 5 4, 5, 6	1 1
3	4 5 6	7, 8, 9, 10 9, 10 9, 10	7, 8 9, 10 9, 10	1 2 2
4	7 8	9, 10 9, 10	9, 10 9, 10	1 1
5	9 10	∅ ∅	∅ ∅	

(4). Draw the graph of hierarchy structure.

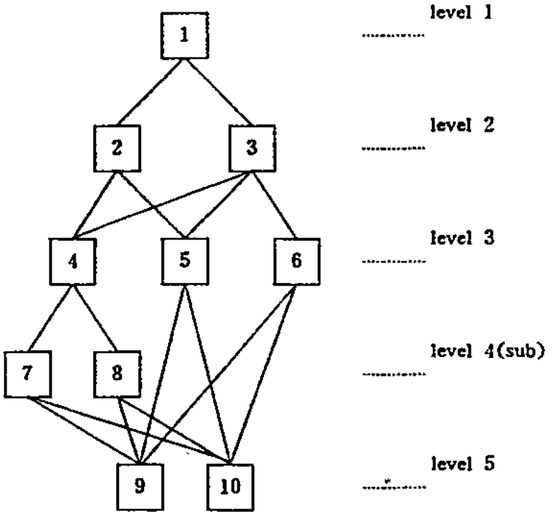


Fig 4

Ⅳ. Conclusion

This paper has presented a method to obtain a hierarchy structure from a directed graph. It will be useful for a complicated system. Its software is being made. Also the method can be extended to deal with backfeed structures. It remains for future research.

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