ISAHP 1999, Kobe, Japan, August 12-14, 1999

Model based AHP

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Abstract: Sekitani and Yamaki introduced two new concepts of *self-evaluation value* and *non-self-evaluation value* into AHP and showed that the eigenvalue method can be formulated as some mathematical programming problems with the ratios of the self-evaluation value to the non-self-evaluation value. This study develops a new discrepancy-minimization problem with the ratios of the self-evaluation value to the non-self-evaluation value and their reciprocals. We show that its optimal solution is identical to the principal eigenvectors of the pairwise comparison matrix. We compare the analytical properties of the proposed optimization model with that of Harker method for the case of AHP with incomplete information.

1 Introduction

The eigenvalue method (EM), that is to find a principal eigenvector of a pairwise comparison matrix, is widely used in AHP. It has been suggested that Frobenius' theorems play an important role to guarantee the existence and the uniqueness of the weight vector which is provided by EM.

Recently Sekitani and Yamaki (Sekitani and Yamaki,1999) focused on Frobenius' min-max theorem from the viewpoint of mathematical programming and then introduced two new concepts of *self-evaluation value* and *non-self-evaluation value* into AHP. The i^{th} self-evaluation value is the i^{th} component w_i of the weight vector w and the i^{th} non-self-evaluation value is $(a_iw - w_i)/(n-1)$, where a_i is the i^{th} row vector of the pairwise comparison matrix of order n. They showed that Frobenius' min-max theorem is interpreted as the following two optimization models P_1 and P_2:

$$(P_1) \qquad \max_{w>0} \min \left\{ \frac{a_1w - w_1}{(n-1)w_1}, \ldots, \frac{a_nw - w_n}{(n-1)w_n} \right\}$$

$$(P_2) \qquad \min_{w>0} \max \left\{ \frac{a_1w - w_1}{(n-1)w_1}, \ldots, \frac{a_nw - w_n}{(n-1)w_n} \right\}$$

The model P_1 minimizes the largest ratio of w_i to $(a_iw - w_i)/(n-1)$ and the model P_2 maximizes the least ratio of w_i to $(a_iw - w_i)/(n-1)$. Sekitani and Yamaki also proved that the optimal solutions of both P_1 and P_2 are identical and equivalent to a principal eigenvector of the pairwise comparison matrix.

In this study, in order to combine the two optimization models P_1 and P_2, we develop a new discrepancy-minimization problem that evaluates the ratios of w_i to $(a_iw - w_i)/(n-1)$ and their reciprocals.

AHP assumes that all pairs of alternatives/objects should be compared. Therefore, AHP with incomplete pairwise comparisons is an exceptional case that needs a special method, e.g., Harker method (Harker,1987). In this study, we apply the above three models (P_1, P_2 and the combined one) to the case of AHP with incomplete information as a natural extension of the models. We show that these three models can deal with AHP with incomplete pairwise comparisons as well as that with all pairwise comparisons.

From the graph-theoretical argument we discuss the analytical properties of these three optimization models through the comparisons with Harker method.

*The author was partially supported by Grand-in-Aid for Scientific Research of the Ministry of Education, Science, Sports and Culture, Grant No. 11780328 and Shizuoka University fund for engineering research.

2 Fundamental Theorems and Generalization of Key Concepts

AHP with all pairwise comparisons provides the pairwise comparison matrix A whose (i, j) entry is the pairwise comparison value a_{ij} of the i^{th} alternative/object and the j^{th} alternative/object. Because the pairwise comparison value a_{ij} is defined positive, the comparison matrix A is positive and irreducible. Therefore the following well-known theorem guarantees that the weight vector by EM is unique, and that it is positive.

Theorem 1 (Perron-Frobenius' Theorem (Takayama,1985)) *Suppose that A is an irreducible nonnegative matrix. Then there are an eigenvalue λ and the corresponding eigenvector w satisfying the following two conditions:*

(1) $Aw = \lambda w$, $\lambda > 0$, $w > 0$ *and* $\lambda \geq |\alpha|$ *for every eigenvalue α of the matrix A.*

(2) *λ is a single root of the characteristic equation of A.*

Furthermore Frobenius' min-max theorem states that the two mathematical programming problems (stated below) have the same optimal solution and that it is identical to the weight vector of EM.

Theorem 2 (Frobenius' min-max Theorem (Furuya,1957)) *Suppose that A is a nonnegative matrix of order n, and that $\lambda_{\max}$ is the principal eigenvalue of A. Then for every n-dimensional positive vector w,*

$$\min\left\{\frac{a_1 w}{w_1}, \ldots, \frac{a_n w}{w_n}\right\} \leq \lambda_{\max} \leq \max\left\{\frac{a_1 w}{w_1}, \ldots, \frac{a_n w}{w_n}\right\}. \tag{1}$$

Furthermore, if the matrix A is irreducible,

$$\max_{w>0}\min\left\{\frac{a_1 w}{w_1}, \ldots, \frac{a_n w}{w_n}\right\} = \lambda_{\max} = \min_{w>0}\max\left\{\frac{a_1 w}{w_1}, \ldots, \frac{a_n w}{w_n}\right\}, \tag{2}$$

where the two equalities in (2) hold for every positive eigenvector w corresponding to the principal eigenvalue $\lambda_{\max}$.

Sekitani and Yamaki proposed the model-based AHP with all pairwise comparisons of n alternatives as follows: the model-based AHP supposes that every alternative evaluates itself, and that it gives itself a positive real number. Let w_i be the positive real number given to the i^{th} alternative by itself. The value w_i of the i^{th} alternative is called the i^{th} self-evaluation value. The value $a_{ij}w_j$ represents the evaluation value of the i^{th} alternative from the viewpoint of the j^{th} alternative when the j^{th} self-evaluation value is w_j. It is called external evaluation of i by Takahashi (Takahashi,1999). Since the number of the alternatives is n, the number of the evaluation values of the i^{th} alternative from the viewpoint of others is $n-1$. Averaging $a_{ij}w_j$ over j except i, we obtain $(\sum_{j\neq i} a_{ij}w_j)/(n-1)$ This we can call the "averaging principle." Because $a_{ii} = 1$, we call $(a_i w - w_i)/(n-1)$ the i^{th} non-self-evaluation value.

For AHP with the incomplete pairwise comparisons, let a_{ij} be the pairwise comparison value when the pair of the alternatives i and j is evaluated by a decision maker, and let a_{ij} be 0 when the pair of the alternatives i and j is not evaluated. Let $a_{ii} = 1$ for $i = 1, \ldots, n$ and $a_{ji} = 1/a_{ij}$ for $a_{ij} > 0$. Then the nonnegative matrix $A = (a_{ij})$ is well defined. We call $A = (a_{ij})$ an incomplete pairwise comparison matrix. We then define K_i as the number of the positive off-diagonal element a_{ij} for $i = 1, \ldots, n$.

As in the case of complete information, w_i is the i^{th} self-evaluation value and $(a_i w - w_i)/K_i$ is called the i^{th} non-self-evaluation value. This definition of non-self-evaluation value is a natural extension of the complete information case, because in this case we have $K_i = n - 1$ for $i = 1, \ldots, n$. Hence the definitions of the self-evaluation value and the non-self-evaluation value in AHP with incomplete information include those in AHP with complete information.

3 Optimization Models for the Incomplete Information Case

This section discusses the incomplete information case in the model-based AHP which is based on the self-evaluation and the non-self-evaluation. The complete information case is dealt as a special case of the incomplete information case.

By introducing the generalized definitions of the self-evaluation value w_i and the non-self evaluation value $(a_i w - w_i)/K_i$ into P_1 and P_2, we formulate the following discrepancy minimization problems with the ratios of the self-evaluation value to the non-self-evaluation value:

$$(Q_1) \qquad \max_{w>0} \min \left\{ \frac{a_1 w - w_1}{K_1 w_1}, \ldots, \frac{a_n w - w_n}{K_n w_n} \right\}$$

$$(Q_2) \qquad \min_{w>0} \max \left\{ \frac{a_1 w - w_1}{K_1 w_1}, \ldots, \frac{a_n w - w_n}{K_n w_n} \right\}$$

P_1/P_2 is identical to Q_1/Q_2 with the complete information, that is $K_i = n-1$ for $i = 1, \ldots, n$. For the matrix A, we define the i^{th} row vector $\hat{a}_i = (a_i - e_i)/K_i$ for $i = 1, \ldots, N$, where e_i is the i^{th} unit row vector. The matrix of $\hat{a}_i$ is then defined as:

$$\hat{A} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \end{bmatrix}. \tag{3}$$

Lemma 3 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Then $\hat{A}$ is also nonnegative and irreducible.*

Proof: Since $\hat{a}_{ii} = (a_{ii} - 1)/K_i = 0$ and $\hat{a}_{ij} = a_{ij}/K_i$, $\hat{A}$ is nonnegative and irreducible. □

Lemma 4 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Then the principal eigenvalue of $\hat{A}$ is a single root of its characteristic equation and there exists a positive principal eigenvector of $\hat{A}$.*

Proof: It is directly followed from Lemma 3 and Theorem 1. □

The following two theorems state the relationship between a principal eigenvector of $\hat{A}$ and an optimal solution of Q_1 or Q_2.

Theorem 5 *Suppose that A is nonnegative and irreducible. Let v be any positive n-dimensional vector other than a principal eigenvector of A, then*

$$\min \left\{ \frac{a_1 v}{v_1}, \ldots, \frac{a_n v}{v_n} \right\} < \lambda_{\max} < \max \left\{ \frac{a_1 v}{v_1}, \ldots, \frac{a_n v}{v_n} \right\}, \tag{4}$$

where $\lambda_{\max}$ is the principal eigenvalue of A.

Proof: We denote the transpose operator for a matrix or a vector by T. Let $\lambda_{\max}$ be the principal eigenvalue of A. Then, the principal eigenvalue of A^T is $\lambda_{\max}$ and there exists a positive principal vector u of A^T corresponding to $\lambda_{\max}$ since A^T is also nonnegative and irreducible.

We will consider the two assumptions $a_i v/v_i \leq \lambda_{\max}$ for every $i = 1, \ldots, n$ and $a_i v/v_i \geq \lambda_{\max}$ for every $i = 1, \ldots, n$, and lead to contradiction under either assumption.

First suppose that $a_i v/v_i \leq \lambda_{\max}$ for every $i = 1, \ldots, n$. Since v is not a principal eigenvector of A, there exists an index l such that $a_l v/v_l < \lambda_{\max}$. It follows from the positiveness of v that

$$Av \leq \lambda v \quad \text{and} \quad a_l v < \lambda v_l \ \text{ for some } l.$$

This means from $u^T A = \lambda_{\max} u^T$ that

$$0 > \sum_{i=1}^{n} u_i (a_i v - \lambda_{\max} v_i) = \sum_{i=1}^{n} u_i a_i v - \lambda_{\max} u_i v_i = \lambda_{\max} u^T v - \lambda_{\max} u^T v = 0,$$

which is contradiction.

The other assumption also leads to contradiction in the same manner. □

Theorem 6 *Suppose that an incomplete pairwise comparison matrix A is irreducible. An optimal solution of Q_1 is equal to a positive principal eigenvector of $\hat{A}$, and vice versa. An optimal solution of Q_2 is also equal to a positive principal eigenvector of $\hat{A}$, and vice versa.*

Proof: It follows from Lemma 3 that $\hat{A}$ is nonnegative and irreducible. Let $\hat{\lambda}_{\max}$ be the principal eigenvalue of $\hat{A}$. Then it follows from (2) of Theorem 2 that

$$\max_{w>0} \min \left\{ \frac{\hat{a}_1 w}{w_1}, \ldots, \frac{\hat{a}_n w}{w_n} \right\} = \hat{\lambda}_{\max} = \min_{w>0} \max \left\{ \frac{\hat{a}_1 w}{w_1}, \ldots, \frac{\hat{a}_n w}{w_n} \right\}. \tag{5}$$

Since $\hat{a}_i w = (a_i - e_i)w/K_i = (a_i w - w_i)/K_i$, the left hand and the right hand of (5) are equivalent to Q_1 and Q_2, respectively. Hence it follows from Theorem 2 that a positive principal eigenvector of $\hat{A}$ is an optimal solution of both Q_1 and Q_2.

Let v be any positive vector other than a principal eigenvector of $\hat{A}$. Since $\hat{A}$ is nonnegative and irreducible, it follows from Theorem 5 that

$$\begin{aligned} \min \left\{ \frac{\hat{a}_1 v}{v_1}, \ldots, \frac{\hat{a}_n v}{v_n} \right\} &= \min \left\{ \frac{a_1 v - v_1}{K_1 v_1}, \ldots, \frac{a_n v - v_n}{K_n v_n} \right\} < \hat{\lambda}_{\max} \\ &< \max \left\{ \frac{\hat{a}_1 v}{v_1}, \ldots, \frac{\hat{a}_n v}{v_n} \right\} = \max \left\{ \frac{a_1 v - v_1}{v_1}, \ldots, \frac{a_n v - v_n}{v_n} \right\}. \end{aligned}$$

Therefore v is not an optimal solution of either Q_1 nor Q_2. □

In order to combine the two optimization problems Q_1 and Q_2, we propose the following discrepancy minimization problem that evaluates the ratios of the self-evaluation value to non-self-evaluation value and their reciprocal:

$$(Q_3) \qquad \min_{w>0} \max \left\{ \frac{a_1 w - w_1}{K_1 w_1}, \ldots, \frac{a_n w - w_n}{K_n w_n}, \frac{K_1 w_1}{a_1 w - w_1}, \ldots, \frac{K_n w_n}{a_n w - w_n} \right\}$$

Lemma 7 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Then Q_3 has an optimal solution.*

Proof: Let $\hat{\lambda}_{\max}$ be the principal eigenvalue of $\hat{A}$. Then it follows from Theorem 2 that for every positive vector v

$$\max \left\{ \frac{a_1 v - v_1}{K_1 v_1}, \ldots, \frac{a_n v - v_n}{K_n v_n}, \frac{K_1 v_1}{a_1 v - v_1}, \ldots, \frac{K_n v_n}{a_n v - v_n} \right\} \geq \max \left\{ \frac{a_1 v - v_1}{K_1 v_1}, \ldots, \frac{a_n v - v_n}{K_n v_n} \right\} \geq \hat{\lambda}_{\max}$$

and

$$\begin{aligned} \max \left\{ \frac{a_1 v - v_1}{K_1 v_1}, \ldots, \frac{a_n v - v_n}{K_n v_n}, \frac{K_1 v_1}{a_1 v - v_1}, \ldots, \frac{K_n v_n}{a_n v - v_n} \right\} &\geq \max \left\{ \frac{K_1 v_1}{a_1 v - v_1}, \ldots, \frac{K_n v_n}{a_n v - v_n} \right\} \\ &= \left(\min \left\{ \frac{a_1 v - v_1}{K_1 v_1}, \ldots, \frac{a_n v - v_n}{K_n v_n} \right\} \right)^{-1} \geq \frac{1}{\hat{\lambda}_{\max}}. \end{aligned}$$

Let w be a positive principal eigenvector of $\hat{A}$. Then we have

$$\begin{aligned} \max \left\{ \frac{a_1 w - w_1}{K_1 w_1}, \ldots, \frac{a_n w - w_n}{K_n w_n}, \frac{K_1 w_1}{a_1 w - w_1}, \ldots, \frac{K_n w_n}{a_n w - w_n} \right\} &= \max \left\{ \hat{\lambda}, \frac{1}{\hat{\lambda}_{\max}} \right\} \\ &\leq \max \left\{ \frac{a_1 v - v_1}{K_1 v_1}, \ldots, \frac{a_n v - v_n}{K_n v_n}, \frac{K_1 v_1}{a_1 v - v_1}, \ldots, \frac{K_n v_n}{a_n v - v_n} \right\} \end{aligned}$$

for every positive vector v. □

Theorem 8 *Suppose that an incomplete pairwise comparison matrix A is irreducible. Let $\hat{\lambda}_{\max}$ be the principal eigenvalue of $\hat{A}$, then the optimal values of Q_3 is $\max\left\{\hat{\lambda}_{\max}, \hat{\lambda}_{\max}^{-1}\right\}$. Furthermore an optimal solution of Q_3 is equal to a positive principal eigenvector of $\hat{A}$, and vice versa.*

Proof: The optimal value is already shown in the proof of Lemma 7. Therefore, we only have to show that an optimal solution of Q_3 is a positive principal eigenvector of $\hat{A}$.

Suppose that v is any positive vector other than a positive principal eigenvector of $\hat{A}$. Then it follows from Theorem 5 that $(\min\{(a_1v - v_1)/K_1v_1, \ldots, (a_nv - v_n)/K_nv_n\})^{-1} > \hat{\lambda}_{\max}^{-1}$ and that $\max\{(a_1v - v_1)/K_1v_1, \ldots, (a_nv - v_n)/K_nv_n\} > \hat{\lambda}_{\max}$. This means that

$$\max\left\{\frac{a_1v - v_1}{K_1v_1}, \ldots, \frac{a_nv - v_n}{K_nv_n}, \frac{K_1v_1}{a_1v - v_1}, \ldots, \frac{K_nv_n}{a_nv - v_n}\right\} > \max\left\{\hat{\lambda}_{\max}, \frac{1}{\hat{\lambda}_{\max}}\right\}.$$

□

Theorem 8 asserts that Q_1, Q_2 and Q_3 have the same optimal solutions.

4 Some Properties of the Optimization Models

In order to describe the structure of the incomplete pairwise comparisons for n alternatives, we consider the following undirected graph with n nodes: If a pair (i, j) of alternatives i and j is compared by a decision maker, the arc (i, j) between the node i and the node j is defined. We denote the graph corresponding to the incomplete pairwise comparison matrix A by $G(A)$. In the case of the incomplete information, the graph is not complete.

Harker method is available for evaluating the weight vector from an irreducible incomplete pairwise matrix A of order n and the weight vector of Harker method is a principal eigenvector of A with the diagonal entry a_{ii} replaced by $n - K_i$. Therefore we formulate the following optimization problem corresponding to Harker method:

$$(Q_4) \qquad \min_{w>0} \max\left\{\frac{a_1w}{w_1} + n - K_1 - 1, \ldots, \frac{a_nw}{w_n} + n - K_n - 1\right\}.$$

Lemma 9 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. An optimal solution of Q_4 is equal to a principal eigenvector of A with the diagonal entry a_{ii} replaced by $n - K_i$, and vice versa.*

The proof is omitted.

Theorem 10 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. Assume that $K_1 = \cdots = K_n$. An optimal solution of Q_1, Q_2 and Q_3 is equal to an optimal solution of Q_4, and vice versa.*

Proof: From theorems 6 and 8, the optimal solutions of Q_1, Q_2 and Q_3 are equivalent, we only have to show one of them is equivalent to Q_4.

If $K_1 = \cdots = K_n$, then it follows that

$$\begin{aligned}
\min_{w>0} \max\left\{\frac{a_1w - w_1}{K_1w_1}, \ldots, \frac{a_nw - w_n}{K_nw_n}\right\} &= \frac{1}{K_1} \min_{w>0} \max\left\{\frac{a_1w - w_1}{w_1}, \ldots, \frac{a_nw - w_n}{w_n}\right\} \\
&= \frac{1}{K_1} \min_{w>0} \max\left\{\frac{a_1w}{w_1} - 1, \ldots, \frac{a_nw}{w_n} - 1\right\} \\
&= \frac{1}{K_1} \min_{w>0} \max\left\{\frac{a_1w}{w_1}, \ldots, \frac{a_nw}{w_n}\right\} - \frac{1}{K_1}.
\end{aligned}$$

Hence an optimal solution of Q_2 is equal to an optimal solution of $\min_{w>0} \max\{a_1w/w_1, \ldots, a_nw/w_n\}$. Furthermore since $K_1 = \cdots = K_n$, Q_4 is equivalent to

$$\min_{w>0} \max\left\{\frac{a_1w}{w_1}, \ldots, \frac{a_nw}{w_n}\right\} + n - K_1 - 1.$$

Therefore an optimal solution of Q_4 is equal to $\min_{w>0} \max\{a_1w/w_1, \ldots, a_nw/w_n\}$. □

All nodes of the graph $G(A)$ have the same degree if and only if $K_1 = \cdots = K_n$. Such a graph is called regular. The above theorem can be also expressed in terms of graphs:

Corollary 11 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that $G(A)$ is connected and regular. An optimal solution Q_4 is equal to an optimal solution of Q_1, Q_2 and Q_3, respectively, and vice versa.*

The following theorem guarantees that both Q_3 and Q_4 provide non-biased weights for the consistent pairwise comparison values.

Theorem 12 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. Assume that the optimal value of Q_4 is n. An optimal solution of Q_4 is equal to an optimal solution of Q_3.*

Proof: Suppose that w is any optimal solution of Q_4. Since A with all diagonal entries a_{ii} replaced by $n-K_i$ is a nonnegative irreducible matrix and the optimal value of Q_4 is n, it follows from Theorem 2 that $n = a_1w/w_1 + n - K_1 - 1 = \cdots = a_nw/w_n + n - K_n - 1$. Therefore we have $(a_iw - w_i)/w_i = K_i$ for every $i = 1, \ldots, n$. This means form Theorem 8 that

$$\begin{aligned}\max\left\{\hat{\lambda}_{\max}, \frac{1}{\hat{\lambda}_{\max}}\right\} &\leq \max\left\{\frac{a_1w - w_1}{K_1w_1}, \ldots, \frac{a_nw - w_n}{K_nw_n}, \frac{K_1w_1}{a_1w - w_1}, \ldots, \frac{K_nw_n}{a_nw - w_n}\right\} \\ &= 1 \leq \max\left\{\hat{\lambda}_{\max}, \frac{1}{\hat{\lambda}_{\max}}\right\},\end{aligned}$$

where $\hat{\lambda}_{\max}$ is the principal eigenvalue of $\hat{A}$. Therefore w is an optimal solution of Q_3. □

Corollary 13 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. Assume that the optimal value of Q_3 is 1. An optimal solution of Q_3 is equal to an optimal solution of Q_4.*

Proof: It is trivial from the proof of Theorem 12. □

The above two assertions means from Theorem 8 that an optimal solution of Q_4 is an optimal solution of Q_i for $i = 1, 2, 3$, respectively.

Corollary 14 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that it is irreducible. Assume that the optimal value of Q_4 is n. An optimal solution of Q_4 is equal to an optimal solution of Q_i for $i = 1, 2, 3$, respectively.*

Here, we consider the special structure of $G(A)$, a spanning tree.

Corollary 15 *Suppose that A is an incomplete pairwise comparison matrix of order n, and that $G(A)$ is a spanning tree. An optimal solution of Q_4 is equal to an optimal solution of Q_i for $i = 1, 2, 3$, respectively.*

Proof: If $G(A)$ is a spanning tree, the rank of the node-arc incidence matrix of $G(A)$ is $n-1$. Therefore there exists a positive vector $w = (w_1, \ldots, w_n)^T$ such that $K_i\hat{a}_{ij} = w_i/w_j$ for $\hat{a}_{ij} > 0$ and we have

$$\frac{\hat{a}_iw}{w_i} = \frac{a_iw - w_i}{K_iw_i} = \frac{\sum_{a_{ij}>0} w_i - w_i}{K_iw_i} = \frac{(K_i + 1)w_i - w_i}{K_iw_i} = 1$$

for every $i = 1, \ldots, n$. Hence the optimal value of Q_3 is 1, which implies by Corollary 13 that the optimal value of Q_4 is n. This assertion is held by Corollary 14. □

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