THE REVISION OF THE JUDGMENT MATRIX

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ABSTRACT

For the eigenvector priority method , we know that some elements of a judgment matrix would be revised if the principal eigenvalue of the matrix is too great. T.L.Saaty [1] suggested some revision methods. This paper discusses the problem in theory. For a situation that one pair of elements are revised only, we show that which elements , how are revised will lead a desent of the principal eigenvalue. For the repeated revision, we construct a iterative procedur and prove its convergence.

1. INTRODUTION

The eigenvetor priority method of the single criteria needs to evaluate the principal eigenvalue and eigenvector of a judgment matrix. If evaluated principal eigenvalue is too great, it shows that the consistency degree of the judgment matrix is bad, and so that the reliability of the obtained priority vector is bad also. In this situation, we should revise some elements of the judgment matrix. T.L.Saaty [1] suggested some revision methods. This paper discusses a situation which revise only one pair of elements. We show that which elements, how are revised will lead a desent of the principal eigenvalue. For the repeated revision, We construct a iterative procedure and prove its convergence. Just as pointed as by T.L.Saaty [1], to realize the revision of a judgment matrix should pass through the revision of the practice judgment. However, reseaches here is meaningful.

2. MAIN RESULTS

Let $A=[a_{ij}]$ be an $n\times n$ positive reciprocal matrix , $\rho(A)$ denotes the principal eigenvalue (Perron root) of the matrix A. a_{uv} be any nondiagonal element in A, we change a_{uv} and a_{vu} into ta_{uv} and a_{vu}/t respectively where $t>0$ and other elements do not change. Thus, we obtian a new matrix $A_{pq}(t)$, say. If $\rho(A_{pq}(t))>\rho(A)$ is always tenable for any $t\neq 1$, we say that the element a_{pq} is proper. Conversely, if there exists some or other t such that $\rho(A_{pq}(t))<\rho(A)$, we say that a_{pq} is improper. It is quite evident that a_{pq} be proper is equivalent to that a_{qp} be proper. We have the following results.

THEOREM 1. Let $A=[a_{ij}]$ be an $n\times n$ positive reciprocal matrix.

(1) Assume $w=(w_1, w_2, \cdots, w_n)^T$ and $u=(u_1, u_2, \cdots, u_n)^T$ are, respectively, the right and left principal eigenvector of the matrix $A=[a_{ij}]$, and

$$g_{ij}=(a_{ij}w_j/w_i)(a_{ij}u_i/u_j) \qquad (2\text{-}1)$$

then, a_{ij} be proper is equivalent to $g_{ij}=1$.

(2) If a_{pq} is improper, then there exists two positive numbers t_1 and t_2, $t_1<t_2$ and $t_1=1$ or $t_2=1$ such that

$$\rho(A_{pq}(t))<\rho(A) \quad \text{when} \quad t\in(t_1, t_2);$$

$$\rho(A_{pq}(t))=\rho(A) \quad \text{when} \quad t=t_1 \text{ or } t=t_2;$$

$$\rho(A_{pq}(t))>\rho(A) \quad \text{when} \quad t<t_1 \text{ or } t>t_2.$$

(3) If $g_{pq}>1$, then $t_1<1/g_{pq}$, $t_2=1$; If $g_{pq}<1$, then $t_1=1$, $1/g_{pq}<t_2$.

A positive reciprocal matrix is called consistent if $a_{ij}a_{jk}=a_{ik}$ for all i, j, k.

THEOREM 2. A positive reciprocal matrix is consistent if and only if its all nondiagonal elements are proper.

The theorem 2 shows that a nonconsistent positive reciprocal matrix must have some improper elements. Using (1) of the theorem 1, we can test which elements are improper. Moreover, (2) and (3) of the theorem 1 supply a revision direction for to decrease the principal eigenvalue.

Because to obtain a satisfied result is not certain when we revise a pair of elements one time, therefore repeated revisions are needful. On this, we propose a iterative algorithm.

ALGORITHM I. Let $A=[a_{ij}]$ be an $n\times n$ nonconsistent positive reciprocal matrix, a matrix sequence $\{A(k)\}$ is produced by the following iterative procedure.

1. Let $A(1)=A$ and $k=1$.

2. For matrix $A(k)$, compute its right and left principal eigenvector respctively

$$w(k)=(w_1(k), \cdots, w_n(k)) \text{ and } u(k)=(u_1(k), \cdots, u_n(k)).$$

and let

$$g_{ij}(k)=[a_{ij}(k)w_j(k)/w_i(k)]\,[a_{ij}(k)u_i(k)/u_j(k)]$$

3. If $g_{ij}(k)=1$ for all i,j, then algorithm ends. At present , the matrix $A(k)$ is consistent.

4. Firstly, choose (r,s) satisfying $g_{rs}(k)=\max_{i,j}\{g_{ij}(k)\}$

and choose arbitrarily a number $t(k)$ in the interval $[1/g_{rs}(k), 1-\varepsilon(1-1/g_{rs}(k))]$, where the positive number $\varepsilon<1$. Next let

$$a_{ij}(k+1) = \begin{cases} t(k)a_{rs}(k), & (i,j)=(r,s) \\ a_{sr}(k)/t(k), & (i,j)=(s,r) \\ a_{ij}(k), & (i,j)\neq(r,s),(s,r) \end{cases}$$

5. Let $k=k+1$ and go to step 2.

If the algorithm I ends in the k-th iteration, then the produced $A(k)$ is consistent. oherwise, it produces a matrix sequence $\{A(K)\}$. On this we have the following convergent theorem.

THEOREM 3. Assume $A=[a_{ij}]$ be an $n\times n$ positive reciprocal matrix which is non-consistent, and a matrix sequence $\{A(k)\}$ is produced by the algorithm I', then we have

$$\rho(A(k+1))<\rho(A(k)) \quad \text{for all } k.$$

and

$$\lim_{k\to\infty}\rho(A(k))=n.$$

3. PROOF OF THE THEOREMS

At first, we prove two lemmas.

LEMMA 1. Assume the definitions of matrixs A and $A_{1n}(t)$ are as section 2. Simply denote $\lambda=\rho(A)$. Let $A_1(t)$ be the costructed submatrix by the former $(n-1)$ rows of $A_{1n}(t)$. Suppose that $x=(x_1, \cdots, x_n)^T$ is the solution of the system of quations.

$$A_1(t)x=\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \quad x_n=1 \qquad (3-1)$$

In addition, let

$$f(t)=a_{n1}x_1 t+\sum_{j=2}^{n-1}a_{nj}x_j+(1-\lambda)x_n \tag{3-2}$$

Then we have that $\rho(A_{1n}(t))<\lambda$ $(=\lambda, >\lambda)$ if $f(t)<0$ $(=0,>0)$.

PROOF. Let the produced submatrix by the former n-1 columns of $A_1(t)$ be A2. It is well known that $\rho(A_2)<\lambda$ ([3], p30), consequently, $(\lambda I-A_2)$ is invertible and

$$(\lambda I-A_2)^{-1}=(1/\lambda)\left(\sum_{k=0}^{\infty}(A_2/\lambda)^k\right)>0$$

Therefore, the solution of the system of equations (3-1)

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = (\lambda I-A_2)^{-1}\begin{bmatrix} ta_{1n} \\ a_{2n} \\ \vdots \\ a_{n-1\,n} \end{bmatrix}; \quad x_n=1 \tag{3-3}$$

is a positive vector.

Let $X=\mathrm{diag}\{x_1,x_2,\cdots,x_n\}$ and $B=[b_{ij}]=X^{-1}A_{1n}(t)X$ Thus from (3-1) we have

$$\sum_{j=1}^{n}b_{ij}=\lambda \quad i=1,2,\cdots,n-1 \tag{3-4}$$

from (3-2), it follows that

$$f(t)=\sum_{j=1}^{n}b_{nj}-\lambda \tag{3-5}$$

therefore, as $f(t)<0$ $(=0,>0)$ so

$$\sum_{j=1}^{n}b_{nj}<\lambda \quad (=\lambda, >\lambda) \tag{3-6}$$

using lemma 2.5 of [3], from (3-4) and (3-6) we conclude that

$$\rho(B)=\rho(A_{1n}(t))<\lambda \qquad (=\lambda, >\lambda) \tag{3-7}$$

which completes the proof.

LEMMA 2. The difinition of the function f(t) is the same as the lemma 1, we have that

1 If f(t) has the double roots t-1, then element a_{1n} is proper

2 If f(t) has two distinct positive roots, then element a_{1n} is improper

PROOF Let $H=(h_{ij})=(\lambda I-A_n)^{-1}$, now the expression of the solution (3-3) is

$$x_i = t h_{i1} a_{1n} + \sum_{j=2}^{n-1} h_{ij} a_{jn}, \quad i=1,2,\cdots,n-1 \qquad (3\text{-}8)$$

From (3-8) and (3-2), we reduce that

$$f(t)=\left(\sum_{j=2}^{n-1} a_{nj} h_{j1} a_{1n}\right)t+\left(\sum_{j=2}^{n-1} a_{n1} h_{1j} a_{jn}\right)/t+\left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1} a_{ni} h_{ij} a_{jn}+h_{11}+1-\lambda\right) \qquad (3\text{-}9)$$

The function f(t) is abbreviated to

$$f(t)=\alpha t+\beta/t+\delta \qquad (3\text{-}10)$$

Because t=1 is the root of f(t), therefore $\delta=-(\alpha+\beta)$.

$$f(t)=\alpha t+\beta/t-(\alpha+\beta) \qquad (3\text{-}11)$$

two roots of the function f(t) are

$$t_1=(\alpha+\beta-|\alpha-\beta|)/2\alpha \qquad (3\text{-}12)$$

$$t_2=(\alpha+\beta+|\alpha-\beta|)/2\alpha$$

namely, either $t_1=1$, $t_2=\alpha/\beta$, if $\beta>\alpha$ or $t_1=\beta/\alpha$, $t_2=1$, if $\beta<\alpha$.

The mininal point of f(t) is $t^*=(\beta/\alpha)^{1/2}$. The condition which f(t) has double roots $t_1=t_2=1$ is $\alpha=\beta$.

If f(t) has double roots, then f(t)>0, i.e. $\rho(A(t))>\lambda$ for any $t\neq1$. consequently, a_{1n} is proper. If f(t) has two distinct roots $t_1<t_2$, then f(t)<0, i.e. $\rho(A(t))<\lambda$ for $t\in(t_1,t_2)$, consequently, a_{1n} is improper, which completes the proof of the lemma 2.

PROOF OF THE THROREM 1

Not lose generalty, we prove the theorem for a_{1n} only. The part 2 of the theorem 1 was proved in the proof of the lemma 2. Below we prove parts 1,3 Let

$$\begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \end{bmatrix} = (\lambda I-A_n)^{-1} \begin{bmatrix} a_{1n} \\ \vdots \\ a_{n-1n} \end{bmatrix}, \quad w_n=1$$

$$(u_1,\cdots,u_{n-1})=(a_{n1},\cdots,a_{nn-1})(\lambda I-A_n)^{-1},\quad u_n=1$$

then $w=(w_1,\cdots,w_n)^T$ and $u=(u_1,\cdots,u_n)^T$ are the right and left principal eigenvector respectively. Using notation $H=[h_{ij}]=(\lambda I-A_n)^{-1}$, we have

$$w_i = \sum_{j=1}^{n-1} h_{ij} a_{jn}$$

$$u_i = \sum_{j=1}^{n-1} a_{nj} h_{ji}$$

and

$$g_{in} = (a_{in} w_n / w_i)(a_{in} u_i / u_n) = (\alpha + h_{ii}) / (\beta + h_{ii}) \qquad (3\text{-}13)$$

where α, β are difinited by (3-9) and (3-10). From the proof of the lemma 2, we know that a_{in} be proper is equivalent to $\alpha = \beta$, and from (3-13), it is equivalent to $g_{in} = 1$.

If $g_{in} > 1$, i.e. $\alpha > \beta$, then $t_i = \beta / \alpha < 1/g_{in} < 1$, $t_n = 1$. similarly, if $g_{in} < 1$, then $t_i = 1$, $t_n = \beta / \alpha > 1/g_{in} > 1$. Thus, the proof is complete.

PROOF OF THE THEOREM 2

The necessity is evident. Prove only the sufficiency. Suppose that nondiagonal elements of the matrix A are all proper. Since $g_{ij} = 1$ for all i, j, we have

$$a_{ij} u_i / w_i = a_{ji} u_j / w_j \qquad \text{for all } i, j \qquad (3\text{-}14)$$

Let $d_i = (u_i / w_i)^{1/2}$, $i = 1, 2, \cdots, n$. and

$$D = \mathrm{diag}(d_1, d_2, \cdots, d_n) \qquad (3\text{-}15)$$

then

$$a_{ij} d_i / d_j = a_{ji} d_j / d_i \qquad \text{for all } i, j \qquad (3\text{-}16)$$

namely

$$DAD^{-1} = D^{-1} A^T D \qquad (3\text{-}17)$$

The above formula shows that DAD^{-1} is a symmetrical matrix, h owever, itself is also positive reciprocal, therefore, $DAD^{-1} = ee^T$, where $e = (1, 1, \cdots, 1)^T$. So that A is consistent, completing the proof of the theorem.

The proof of theorem 3 is omitted.

Finally, we give a example

$$A = \begin{bmatrix} 1 & 2 & 1/2 & 9 \\ 1/2 & 1 & 2 & 9 \\ 2 & 1/2 & 1 & 9 \\ 1/9 & 1/9 & 1/9 & 1 \end{bmatrix}$$

The principal eigenvalue is $\rho(A)=7/3+a$, the right and left principal eigenvector are $w=(1,1,1,a/9)$ and $u=(1,1,1,9a)$ respectively, where $a=((73)^{1/2}-5)/4$. Element $a_{14}=9$ is proper. Moreover,

$$a_{14}-w_1/w_4 = \max_{i,j}\{|a_{ij}-w_j/w_i|\}=1.158$$

This shows that determining revised element by means of $\max_{i,j}\{|a_{ij}-w_j/w_i|\}$ is sometimes not suitable

REFERENCE

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