INCOMPLETE PAIRWISE COMPARISONS WITH THE ANALYTIC HIERARCHY PROCESS: THE METRIC AND ENTHROPY APPROACHES

Kirti Peniwati
PPM Graduate School of Management
Jl. Menteng Raya 9
Jakarta 10340 - Indonesia
kirti@indo.net.id

Abstract

While redundancy in paired comparisons is necessary in the AHP to improve the accuracy of an outcome, it becomes a drawback for its application particularly in a group process. One way to make a group process efficient with the AHP is by reducing the number of judgments. It has been observed that the closer the number of judgments to its maximum, the less efficient the decision making process would be. The idea of incomplete paired comparisons is to provide the minimum number of judgments, then add judgments with the most information value one by one, until a predetermined approximation level is achieved. A method to identify the next judgment to make has been proposed in the literature, using gradients to measure distance between two eigenvectors. Here two alternative approaches are proposed, the first employs Saaty's metric of ratio scales, and the second is based on the concept of entropy used in information theory.

The approach employing Saaty's metric of ratio scales is as follows:

1. Provide (n-1) 'connected judgments';
2. Construct quasireciprocal matrix, Compute eigenvector x and eigenvalue $\lambda_{max;}$
3. Compute gradients $\partial A/\partial x;$
4. Compute 'new' eigenvectors x (within acceptable proportionality of gradients);
5. Compute metrics from the previous x;
6. Rank (missing) entries according to their metrics;
7. If entry with the largest metrics << cut of point, then stop; if not, then:
8. Provide judgment for the missing entry with the largest metric
9. Back to step 2.

The concept of entropy used in information theory suggests an alternative measure for comparing two eigenvectors to determine whether a succeeding one has more information. It takes the following form:

$$E(x_k, x_{k-1}) = \sum_{i=1}^{n} x_{ik} \ln\left(\frac{x_{ik}}{x_{ik-1}}\right)$$

Here $E(x_k, x_{k-1})$ is the relative entropy between x_{k-1} and x_k , where x_{k-1} and x_k are two successive eigenvectors calculated by adding one new judgment and $x_{i,k-1}$ and x_{ik} are their corresponding i^{th} components. This expression can be used to determine the next judgment by taking the latest eigenvector and the modified eigenvectors, produced by the gradient method for each entry, in a similar way as that for the metric.

The proposed approaches is illustrated with an example taken from the AHP literature, producing the following observations and results:

1. The absolute sum of the gradient, the metric, and the relative entropy approaches point to the same entry for the first next judgment to make. The metric and relative entropy approaches practically give identical rank of entries.

2. The additive operation of the gradients gives rise to the possibility of a negative ratio. We need to make sure that the negative gradient is still relatively small compared with its corresponding element of the most recent eigenvector. However, it should not be too small either, to ensure a faithful outcome. In this case, we may need to calibrate the elements of the gradient vectors. If calibration is required, a percentage must be determined and multiplied to each element of the gradient vector for each entry, to maintain their ratios and their total of zero's.

3. Using the metric approach may generate a problem of having gradients whose elements are relatively large compared with their corresponding eigenvector entries. The sum of gradient approach is not affected by this situation since it uses the gradients directly to rank the entries. The metric method requires one to use the gradients to modify the eigenvector, hence a small gradient element must be ensured. First, it is necessary to calibrate the gradients, and second, to note how sensitive the calibration is to the outcome. In other words, we want to know if calibration is always required, and whether or not we have to use the same calibration factor for each iteration.

4. The gradients have certain properties that need to be considered to determine the method of calibration. The calibration must both maintain the underlying ratios for all the entries, and the total of zeroes for each entry. Multiplying all the elements with the same percentage satisfies these requirements.

5. To examine how sensitive the calibration is, we take the result of the previous iteration and see how the different percentages of gradients affect the ranking of the entries. Our example indicates that taking very small gradients, here from 10% to 30% of its original ones, gives slightly different rankings. Larger gradients, between 40% and 100% of its values, give the same ranking. In this particular example, one cannot go below 40% gradients and may arbitrarily select between 40% to 100% gradients. However, too large a gradient may turn the positive element of the eigenvector into a negative.

6. There is an issue whether one should continue to compute the gradients every time a new judgment is entered. The example indicates that after a certain number of iterations, further iterations will not be necessary since they will give the same rank order. In our example, the third iteration gives the same rank as the second one.

7. Inconsistency may get worse and worse with more judgments, although the overall inconsistency may still be in the acceptable level. In a larger matrix, a constant increase in inconsistency with more judgments may end up at an unacceptable level of inconsistency by the time all the judgments are made. The saving in time would be substantial since following the usual AHP procedure would require the group to make more than the full number of judgments since more judgments may be needed to improve consistency.

8. Analyzing the outcomes of the gradient method using the three different approaches suggest that they might point to the same judgment to make if they are used iteratively. However, using it only once to determine the order of adding judgments may still be acceptable for practical reasons. Our example suggests that after a certain number of iterations, further iterations may not be necessary since they will give the same rank order. In our example, the third iteration gives the same rank as the second one.