THE NONLINEAR NATURE OF PREFERENCES, ITS IMPACT ON THE SENSITIVITY AND EFFECTIVENESS OF MULTIPLE CRITERIA ALTERNATIVES

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ABSTRACT

The mechanisms of creating and processing expert and/or statistical information, which underlie the *AHP/ANP*, do not enable to take into account the nonlinear nature of preferences and their dependence upon intensiveness (or level) of measurable features and qualities of researched and optimized systems. In the traditional methods of multiple criteria (multiobjective) optimization based on the concept of preferences and utilities, the nonlinear nature of preferences as well as the loss of sensitivity and effectiveness of alternatives caused by it can be taken into account, if to proceed from the concave increasing property of a corresponding evaluation function (preference function). If to assume also that the concave increasing preference function $u : E^m \to E^1$ is differentiable on the nonempty convex set of estimates $F_0 \subset E^m$,

$$u(f) - u(\bar{f}) \le \nabla u(\bar{f})^T (f - \bar{f}), \; f \in F_0$$

then from its peculiar differential inequality

$$u(f) = u_0(f) + \nabla u(f)^T f, f \in F_0.$$

one can deduce the decomposition scheme

where $u_0(f)$ is the height of intersection of the hyperplane $H_u = \{(y, f) / y = u(\bar{f}) + \nabla u(\bar{f})^T (f - \bar{f}), \; f \in F_0\}$, tangential to the surface of the function $u(f)$, with the axle of the latter at the point $\bar{f}$, and $\nabla u(\bar{f})$ is the gradient of the function. The second component of the decomposition, $v(f) = \nabla u(f)^T f$, is related to the relative sensitivity function $\sigma(f) \equiv \lim_{t \to 1} \frac{t}{u(tf)} \frac{du(tf)}{dt}$ = $\frac{1}{u(f)} \sum_{j=1}^{m} \frac{\partial u(f)}{\partial f_j} f_j$ with the correlation

$$v(f)=\sum_{j=1}^{m}\frac{\partial u(f)}{\partial f_j}f_j=\sigma(f)u(f)$$

("*sensitive*" component $u(f)$). Its items,

$$v_j(f)=\frac{\partial u(f)}{\partial f_j}f_j=\sigma_j(f)u(f),\ j=1,\dots,m$$

, in effect, characterize the relative *weight (importance, return*) of individual criteria and play the same role in alternatives ranking as the eigenvector coordinates $\omega=(\omega_1,\omega_2,\dots,\omega_m)^T$ do in the *AHP/ANP* schemes, but now they depend on the state vector. Their ratio, $v_i(f)/v_j(f)=\sigma_i(f)/\sigma_j(f)=(f_i/f_j)\mu_{ij},\ i,j=1,\dots,\ m,$ where $\mu_{ij},\ i,j=1,\dots,\ m$ are marginal rates of substitution between the criteria, allows to find stable solutions that would satisfy the condition $f_i/f_j=1/\mu_{ij}=\mu_{ji}.$ The maximization of the functions $v(f)$ and $\sigma(f)$ on the surface of $u(f)=const$ generates a trajectory of solutions $f^c(\lambda)$, $\lambda\le 1,$ that satisfy both the maximum of the characteristics $v(f^c)$ and $\sigma(f)$, and a "*consistent*" value of the criteria interaction function $\theta(f^c)=-f^{cT}Hf^c=(1-\lambda)v(f^c)=(1-\lambda)c\sigma(f^c)$, where $\lambda\le 1$ is the Lagrange multiplier, *H* is the Hessian matrix of the function $u(f)$. Also the possibility of using the proposed analytic correlations in the *AHP/ANP* traditional procedures is being discussed. As applications a multiple criteria problem of corporate resources management and diagnostic messages processing in transport systems are being considered.

Keywords: preferences, nonlinear nature, sensitivity, effectiveness of alternatives, optimization on surface, trajectory, stability.

1. Introduction

Applied multiple criteria problems and interactive methods of their solution serve as the context of the proposed study. Traditionally, one of the important streams of scholarship in this field of knowledge was based on the *concept of preferences and utilities* that show up on the set of estimates of alternatives by criteria. Experience and intuition suggest that preferences are of nonlinear nature which leads to the loss of sensitivity and effectiveness of causal relations. The necessity to describe analytically these nonlinear effects and to take them into account in choice and decision making procedures acted as a spur to the present study. Involving the theory of sensitivity allows to construct suitable analytic tools (*measures of relative sensitivity* and *effectiveness* for *nonlinear causal relations*) contributory to ranking solutions according to their preference. The goal is a more adequate description of preferences, raise of accuracy and utility of sought-for solutions and of actions planned on their base.

2. Literature Review

As far as we know, the idea of spatio-temporal dependence of preferences was first stated in (Keeney R., & Raiffa H., 1976). In effect, the logic of constructing effective man – machine methods and procedures for solving practical multiple criteria problems also proceeds from an assumption of dependence of preferences upon an (achieved) level of criteria (Benayoun R., 1971; Hall A.D., 1962; Fandel G., & Wilhelm J., 1976; Geoffrion, A.M., 1972). To develop the ideas of the authors (Geoffrion, A.M., 1972), in (Sarkisyan R., 1992) the possibility was substantiated of representing the relative importance of criteria via the ratio of directional derivatives of criterion functions. The results of some recent studies (DellaVigna S., 2009; Hands W., 2010.) also speak in favour of the dependence of preferences upon the evaluated state and upon trends of its development in time.
The approach contained in the proposal and its analytic correlations (measures of sensitivity and effectiveness, optimization on surface) that characterize the nonlinear preferences are original results of applying the theory of sensitivity to modeling nonlinear effects which show up during the formation of expert information about criteria and alternatives.

3. Objectives and Hypotheses

The main assumption is brought to the existence of the concave increasing twice differentiable preference function on the set of criteria values. Such a function satisfies the differentiable inequality $u(f)-u(\bar{f})\le \nabla u(\bar{f})^T(f-\bar{f}),\ f\in F_0,$ and allows a decomposition in the form $u(f)=u_0(f)+\nabla u(f)^T f$ $=(1-\sigma(f))u(f)+\sigma(f)u(f),\ f\in F_0,$ where $\sigma(f)=\sigma_1(f)+\ldots+\sigma_m(f)$ is the measure of relative sensitivity, and $v(f)=\nabla u(f)^T f = v_1(f)+\ldots+v_m(f)$ is the measure of effectiveness (effective return). The maximization of these characteristics on the preference function level surface allows to reach necessary ranking of criteria values.

4. Research Design

The interactive methods of multiple criteria optimization are in harmony with the spirit of the studies (Saaty T.L., 1994; Saaty T.L., 2001) aimed at finding a *reasonable compromise* between rivaling criteria and requirements. In this sense the hierarchy <*Goal, Criteria, Alternatives*> is inherent to any problem of optimization on the base of multiple criteria logic and serves, according to T. Saaty, as *a mighty speculative model for describing and researching complex systems*.

5. Premises

In one of the earlier studies related to researching the reliability of the computerized system TELEPERM-ME (Siemens) by means of the *AHP/ANP* models we came to a conclusion that the idea of a *"reasonable compromise"* harmonizes with that of *"ideal proportions"* (Geoffrion A., 1972). Such proportions can be obtained by maximizing the functions $v(f)$ and $\sigma(f)$ on the surface of $u(f)=const$. The ratios

$v_i(f)/v_j(f) = \sigma_i(f)/\sigma_j(f) = (f_i/f_j)\mu_{ij},\ \ i, j = 1,\dots,\ m,$ satisfy this quality of solutions. Optimization on the preference function level surface also satisfies Simon's *principle of satisficing* (Simon H., 1979), while the role of consent index in *AHP/ANP* is played by a relation which allows to evaluate the level of the criteria interaction function $\theta(f^c) = -f^{cT}Hf^c = (1-\lambda)v(f^c) = (1-\lambda)c\sigma(f^c).$ This relation is just at all points of the optimal trajectory $f^c(\lambda), \lambda \le 1.$ It means that at every point of the optimal trajectory the level of the criteria interaction function is defined by the maximum value of indicators $v(f)$ and $\sigma(f).$

6. Limitations

It is assumed that during the increase of the criteria values the preference function is aiming at its established value (i.e. it is "*getting sated*"). In this case the measure of its relative sensitivity *σ(f)* is a monotonously decreasing function, while the "sensitive" component *v(f)* reaches its maximum on the level surface of *u(f) = const* and, therefore, the given device can be successfully applied.

According to the Debreu theorem, if the preference relation on the set of criteria values is a *complete preorder* and has properties of *monotonousness* and of *continuity*, then the preference (or utility, value) function can be put in correspondence with it; the property of differentiability is entered for convenience of mathematical analysis. It is not expedient to apply the given approach unless it is assumed that the preference function is "*getting sated*".

7. Conclusion

Our professional interest in the *AHP/ANP* technique and models is conditioned only by issues and prospects of development of interactive methods of applied multiple criteria problems solution; the application aspect is to create decision making support systems for projecting, planning and managing in technical and organizational systems.

In the research practice related to applied multiple criteria problems solution, the interactive methods and procedures play a leading role. The traditional view on the algorithmization of the preferred solutions search process proceeds from the consecutive revelation and description of *preferences* as *a function of state and movement direction in the space of estimates and alternatives*. It is accepted that namely such a multi-step (multi-stage) search process will allow to decrease uncertainty and non-comparableness and to reach a reasonable balancing (Saaty T,L., 2001) between rivaling requirements of criteria. The preferences are of nonlinear nature which generates the loss of sensitivity and effectiveness of a corresponding causal relation. The approach stated in the proposal allows to model more adequately these nonlinear effects and to take them into account while ranking multiple criteria estimates and corresponding alternative solutions. In our view, the relation of these factors to the *AHP/ANP* is obvious.

8. Key References

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9. Applications

The analytical apparatus stated in the proposal can be illustrated by the example of two applications:

a) *Corporate resources management during transport systems maintenance and repair planning*. On the base of regarding the technological relation between the planned work volume and costs of economic factors made for that purpose as a production function $\bar{y} = F(q_1,...,q_n),$ a bicriteria optimization problem is being formulated which includes minimization of the cost function $r(q) = p_1 q_1 + ... + p_n q_n)$ and maximization of the return function $v(q) = \frac{\partial F}{\partial q_1} q_1 + ... + \frac{\partial F}{\partial q_n} q_n.$ The production function, at that, is being regarded as a twice continuously differentiable concave increasing function, thus allowing a decomposition in the form $F(q) = F_0(q) + v(q) = (1 - \sigma(q))F(q) + \nabla F(q)^T q,$ As a suitable preference function on the set of criteria values the approximations $u(f) = a^T f + (1/2) f^T Hf, a + Hf \hbar 0$ and $u(f) = U_0(1 - e^{-\alpha_1 f_1})(1 - e^{-\alpha_2 f_2})$ are being considered, where the coordinates f_1 and f_2 represent corresponding criteria functions. Optimization on the surface $\bar{y} = F(q_1,\ldots,q_n)$ is being made using the Lagrange method.

b) *Diagnostic messages processing for high speed vehicle.*

As it is known, the problem of diagnostics can be considered from the point of view of a problem of choosing one of possible explanations on the base of available data. It is assumed that an observed object is exhaustively described by a set of parameters x_1, $x_2, \ldots, x_n$, which make up the description vector $\mathbf{x} = (x_1, \ldots., x_n)^T \in D$, where D is the space of descriptions, the set of various values of the vector $\mathbf{x}$. Also it is assumed that to every state S_i from among the set of states $S = \{S_1, S_2, \ldots, S_m\}$ corresponds an area $D_i \subset D$, $i = 1, \ldots, m$. If these areas are already established, the problem of diagnostics can be brought to decision making in favour of a state (or hypothesis) S_i any time its description $\mathbf{x}$ falls on an area D_i. As the solution criteria the following are being regarded:

- maximization of the a posteriori probability function $P(S_i/\mathbf{x}) =$„ $i = 1, \ldots, m;$

- minimization of the loss function $l(x \in D_i)$, losses being caused by a false classification.

In case of Euclidean description space, i. e. $D \in E^n$, the criteria are interpreted in terms of corresponding probability density functions on the subsets D_i, $i = 1, \ldots, m$.

In effect' both applications represent a hierarchical model <Goal – Criteria - Alternatives>. In the first application we deal with a continued set of alternatives presented by the level surface equation $\bar{y} = F(q_1,\ldots,q_n)$, while in the second one the set of states S = {S1, S2, … , Sm} subject to identification (or classification) serves as a set of alternatives.

Note that in both cases the preferences depend upon the criteria values. In *AHP/ANP* models the judgment matrix looks as $A = (a_{ij})$, $a_{11} = a_{22} = 1$, $a_{12} = 1/a_{21}$. The corresponding priority estimates will be equal to: $\omega_1 = a_{12}/(1 + a_{12})$, $\omega_2 = 1/(1 + a_{12})$, $\omega_1/\omega_2 = a_{12}$, as follows from the model of numerical judgments processing $A\omega = \lambda_{max}\omega$, $\lambda_{max} = 2$. These estimates of relative priorities are not acceptable for both (or similar) applications, despite the efforts of experts. According to the material of the proposal, their role can be played by the measures $v_i(f)$, $i=1,\ldots,m$, more precisely, by their optimal values.