CONSTRUCTING HIGHLY CONSISTENT PAIRWISE COMPARISON MATRICES IN ANALYTIC HIERARCHY PROCESS (AHP)

Sahika Koyun
Industrial Engineering
Yildiz Technical University
Istanbul, TURKEY
E-mail: skoyun@yildiz.edu.tr

Vildan Cetinsaya Ozkir
Industrial Engineering
Yildiz Technical University
Istanbul, TURKEY
E-mail: cvildan@yildiz.edu.tr

ABSTRACT

AHP provides a decision making framework by quantifying the decision elements in order to evaluate alternative solutions with respect to a specified objective in multiple criteria decision making problems. AHP uses pairwise comparison (PC) data for generating weight vectors of decision elements for final results, which is limited to the consistency of PC matrices. Final results (resulting weight vector) can only be considered as a reliable reflection of the evaluator's opinion, if and only if relevant data is sufficiently consistent. By determining the causes of inconsistency, we develop a new method for constructing highly consistent PC matrices. This study investigates underlying reasons of inconsistency, and explores new tools/methods to derive consistent matrices.

Keywords: inconsistency, pairwise comparison, consistency improvement.

Introduction

PC data is first introduced to decision-making by Saaty (1980), and since used for gathering judgments from individuals. Although PC is an effective tool for gathering data from individuals (mostly experts of a specified area), they tend to be inconsistent due to various reasons. This study focuses mainly on underlying causes of inconsistency and how to construct consistent matrices.

Many researchers (Benítez et al., 2011; Gomez-Ruiz et al., 2009; Siraj et al., 2012; Dadkhah & Zahedi, 1993; Bozóki et al., 2011) focus on how to repair inconsistencies of a matrix, but repairing the consistency rate of a matrix does not necessarily guarantee a valid result of decision process (Gaul, 2012). Rather than repairing consistency rate of a given judgment, the main focus should be how to build a consistent matrix.

There are three main underlying causes of inconsistency stated in the literature. These are can be summarized as: measurement errors, faulty perception of decision maker and the size of PC matrices.

AHP utilizes PC matrices to estimate the weight ratio of two criteria at a time. Let A be the PC matrix for n criterion and a_{ij} is the estimated ratio of the weights of criterion i over criterion j (Saaty,1980)

$$A = [a_{ij}] \qquad \forall\ i,j \in \{1,2,\dots,n\} \tag{1}$$

$$a_{ij} = \frac{w_i}{w_j} \qquad \forall\ i,j \in \{1,2,\dots,n\} \tag{2}$$

If the exact values of w_i and w_j are known or measurable by any kind of measurement unit, the value of a_{ij} can be calculated, exactly. Since w_i and w_j cannot be measured exactly, an estimation of them can be utilized to derive a_{ij}. This estimation causes a deviation from original a_{ij} value, which is denoted as *measurement error* (ε). The estimated value of a_{ij} is defined as (Saaty,1980):

$$a_{ij} = \left(\frac{w_i}{w_j}\right)\varepsilon_{ij} \qquad \forall\ i,j \in \{1,2,\dots,n\} \tag{3}$$

When ε_{ij} is *1*, full consistency occurs. Inconsistencies caused by measurement error can be tolerated up to a limit by setting a consistency threshold. There are several examples of consistency thresholds in the literature. (Aguarón & Moreno-Jiménez, 2002; Alonso & Lamata, 2006; Koczkodaj, 1993; Monsuur, 1997; Vargas, 2007.)

Faulty perception of decision maker's (DM) can also cause inconsistencies in PC matrices. These inconsistent evaluations disrupt the transitivity property by creating cycles in graphical representation. In the literature, the inconsistency caused by measurement error or faulty perception of DM is improved by either revision of existing data or repetition of elicitation process (Saaty, 1980; Benitez et al., 2011; Gomez-Ruiz et al., 2009). Since there is a possibility of the DM giving misleading answers intentionally or has a faulty strict perception on the value of judgments, these repairing processes do not ensure an exact improvement in consistency rate. Instead of repeating the process of elicitation until the positive reciprocal matrix (PRM) becomes consistent, which is also a time consuming method considering the number of inconsistent matrices to be revised; DM can be guided through elicitation process. If any inconsistent judgment is encountered, DM will be guided for an immediate revision. Conclusively, the elicitation process is finalized with highly consistent PC matrices at first attempt, without any further calculation.

Lastly, another cause of inconsistency mentioned in the literature is the amount of PCs to be made. Let *A* be the PRM with order of n, then *total number of PCs to be made* is:

$$\frac{n(n-1)}{2} \qquad N = [1, n]\,, n \in \mathbb{R}^{+} \tag{4}$$

While n increases, the amount of PCs to be made increases as polynomial. Therefore, the effort/time to make these PCs will increase accordingly. AHP enables us to evaluate multiple criteria but it limits DM by setting an upper bound to the total number of criteria. It has been proven that incomplete matrices with specific attributes follow the same rules as the complete ones (Harker, 1987). Then, it is questionable to set an upper bound to the number of criteria.

The graph of a complete PRM is a strongly connected network, while the graph of an incomplete matrix is a spanning tree. Therefore, by using incomplete matrices, missing PCs can be derived from the existing links of these spanning trees. Remembering a tree is a acyclic and a connected graph, the minimum required number of PCs to be made is $(n-1)$ for a PRM with order of n. For a matrix with the order of 9, total number of PCs is 36 while only 8 of them is enough to construct a PRM. Since Saaty (1980) states the upper bound 9 as the total number of criteria to be evaluated at once, the maximum number of judgments will be limited to 36. The main concern should be limiting the total number of PC rather than limiting the total number of criteria to enable consistent evaluations. Hence, the actual limitation should not be on the number of criteria but on the number of PCs. If 36 PCs can be made consistently, then it is inferred that up to 37 criteria can be evaluated consistently with the use of incomplete matrices.

Literature Review

Harker (1987) proves that non-negative, quasi-reciprocal matrices can be used as exactly same manner as positive, reciprocal matrices. Therefore, DM is allowed to answer questions, such as “I do not know” or “I am not sure”, making the questioning process shorter. Therefore, better representation of the responses to certain stimuli may be derived.

Wedley et al. (1993) investigates the effect of different reference items for the first $(n-1)$ PCs. The empirical results show that if the lowest ranked item is utilized as a common reference for the first $(n-1)$ PCs, significantly greater initial accuracy can be achieved.

Ishizaka and Lusti (2004) proposes an expert module to construct consistent matrices. Their module uses the second diagonal of the PRM matrix as the first $(n-1)$ comparisons to eliminate independency of comparisons.

Gaul and Gastes (2012) explain the disadvantages of consistency adjustments for computing acceptable weights for the determination of the underlying overall objective function. They argue that an adjusted matrix, which is derived by a reported matrix, can converge to any consistent matrix. Since the intended consistent matrix is still unknown, it is not possible to investigate if these two consistent matrices are the same.

Hypotheses/Objectives

Ali (1986) shows that the adjacency matrix should be reordered in a decreasing fashion (1st row and column is the team with the most wins, 2nd row and column is the team with the second most wins etc.) in order to create tournament rankings with minimum violations. Wedley et al.'s (1993) results show that the first row should belong to the lowest ranked criterion because estimating the ratios of a_{ij} should be evaluated regarding the lowest ranked criterion, which gives us the first $(n-1)$ comparisons. However

Ishizaka and Lusti (2004) suggests that starting with the first row compromises independency due to psychological reasons.

We propose a new elicitation process that reduces the total number of PC judgments regarding independency and consistency issues. The proposed elicitation process to construct PRM matrices uses a two-step ranking evaluation procedure. Firstly, DM is asked to order rank the set of criteria. In consequence of this ranking process, the lowest ranked criterion is selected as point of origin (c_0). From remaining $(n-1)$ criteria, a randomly chosen criterion (c_1) is compared to the point of origin (the lowest ranked criterion). Process continues by a new randomly chosen criterion (c_2) from a set of remaining $(n-2)$ criteria to be compared to (c_1). When all the criteria are compared, the spanning tree is constructed. By the use of indirect judgments, we can create strongly connected graph (which consists of $\frac{n(n-1)}{2}$ comparisons) of PRM.

To summarize, objectives of this study can be listed as follows:

- To test whether the use of incomplete matrices is statistically as consistent as the use of complete matrices.
- To develop a method that randomizes the first $(n-1)$ comparisons with a specific point of origin.
- To test whether a consistent matrix of greater order than 9 can be created.

Research Design/Methodology

Figure 1 represents the methodology for investigating the construction of consistent PC matrices by defining two hypothesis to examine the applicability of incomplete matrices.

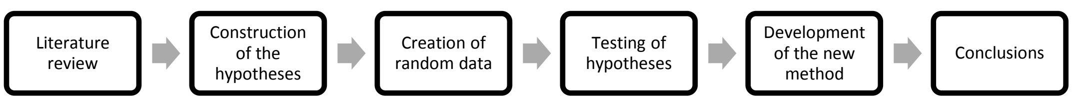

Figure 1 Research design

Data/Model Analysis

For both sets of incomplete and complete matrices, and for each order of matrix from 4 to 11, 10 samples of 1000 matrices were generated with Visual Studio, and their statistical distribution of Consistency Index (CI) is analyzed. The entries for these matrices were selected from the 17 element set (Saaty, 1980), i.e., the integers 1 to 9 and their reciprocals. While constructing the sample set for incomplete matrices, origin point is randomly chosen from n criterion.

In order to test whether the use of incomplete matrices is statistically as consistent as the use of complete matrices, CI distributions of incomplete and complete matrix samples will be analyzed, comparatively.

Hypothesis 1: The number of PCs does not affect the consistency of a PRM

If *Hypothesis 1* is proven to be null, the number of PCs does not have an influence on the consistency of a PRM. In that case, incomplete matrices can be used instead of complete matrices, and *CI* values gathered from this study can be used as *Random Index* (*RI*) *for incomplete matrices* in future studies.

If *Hypothesis 1* is proven to be null, and the use of incomplete matrices is valid, then a method which randomizes the selection of $(n-1)$ PCs with an origin point will be developed.

In order to test whether a matrix of order greater than 9 can be created consistently with the use of incomplete matrices; CI distributions of matrices with order greater than 9 samples will be compared against to samples of matrices with order less than 9.

Hypothesis 2: *PRMs of order greater than 9 can be created consistently with the use of incomplete matrices.*

If *Hypothesis 2* is proven to be null, the order of a PRM does not have an effect on the consistency of matrix. In that case, the upper bound for criteria can be discussed further.

Limitations

To eliminate the effect of point of origin, while creating sample sets for incomplete matrices the point of origin is chosen randomly for each constructed random matrix. The proposed elicitation process constructs a directed spanning sub-graph, and strongly connected graph is derived, consequently. Even though the DM is required to order rank the set of criteria, it is still a possibility that DM's perception remain faulty, and can lead to inconsistencies. However, we hope to minimize the consistency rate by checking each evaluation of DM, instantaneously.

Conclusions

As this study is still ongoing, we believe that the results will support our hypothesis. With the use of incomplete PC matrices, construction of highly consistent matrices will be possible. With the use of proposed process for the construction of PRM matrices, we expect that, causes of inconsistency will be limited to only measurement error.

Key References

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